Holomorphic Automorphisms of Noncommutative Polyballs

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GOALS

- To study free holomorphic functions on regular polyballs and provide analogues of several classical results from complex analysis.
- To obtain a complete description of the group $\mathrm{Aut}(\mathbf{B_n})$ of all free holomorphic automorphisms of the polyball $\mathbf{B_n}$.
- To study the automorphism groups of the operator algebras : $C^*(\mathbf{S})$, $\mathcal{A}_\mathbf{n}$, $\mathbf{F}_\mathbf{n}^\infty$, generated by the left creation operators acting on tensor products of full Fock spaces.

Acknoledgement

- We continue the work of Voiculescu (Lect. Notes Math. 1985), of Davidson and Pitts (Math. Ann. 1998), of Benhida and Timotin (Indiana Univ. Math.J. 2007), of Helton, Klep, McCullough and Singled (JFA 2009), and of P., (Crelle 2010, IMRN 2011).
- In a related context we mention the work of Muhly and Solel (Documenta Math. 2008), and of Power and Solel (JFA 2011).

Noncommutative polyballs

- $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ denotes the set of all tuples $\mathbf{X} = (X_1, \ldots, X_k)$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}$, $s \neq t$.

- The *polyball* :

$$\mathbf{P_n}(\mathcal{H}) := [B(\mathcal{H})^{n_1}]_1 \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1,$$

where $[B(\mathcal{H})^{n_i}]_1$ is the open unit ball

$$\{(X_{i,1}, \ldots, X_{i,n_i}) \in B(\mathcal{H})^{n_i} : \|X_{i,1}X_{i,1}^* + \cdots + X_{i,n_i}X_{i,n_i}^*\| < 1\}.$$

Noncommutative regular polyballs

- The *regular polyball* on the Hilbert space $\mathcal{H}$ is defined by

$$\mathbf{B_n}(\mathcal{H}) := \{\mathbf{X} \in \mathbf{P_n}(\mathcal{H}) : \ \mathbf{\Delta_X}(I) > 0\},$$

  where the *defect mapping* $\mathbf{\Delta_X} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\mathbf{\Delta_X} := \left(id - \Phi_{X_1}\right) \circ \cdots \circ \left(id - \Phi_{X_k}\right),$$

  and $\Phi_{X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y) := \sum_{j=1}^{n_i} X_{i,j} Y X_{i,j}^*, \qquad Y \in B(\mathcal{H}).$$

- *(Abstract) regular polyball*

$$\mathbf{B_n} := \{\mathbf{B_n}(\mathcal{H}) : \ \mathcal{H} \text{ is a Hilbert space}\}.$$

Universal models

- Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i, \ldots, e_{n_i}^i$. The *full Fock space* of H_{n_i} is defined by

$$F^2(H_{n_i}) := \mathbb{C}1 \oplus \bigoplus_{s\geq 1} H_{n_i}^{\otimes s}.$$

- Let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i, \ldots, g_{n_i}^i$ and the identity g_0^i. Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$.

- For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, the left creation operator $S_{i,j}$ on $F^2(H_{n_i})$ is defined by setting

$$S_{i,j} e_\alpha^i := e_j^i \otimes e_\alpha^i, \qquad \alpha \in \mathbb{F}_{n_i}^+.$$

Universal models

Definition

The operator $\mathbf{S}_{i,j}$ acting on $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ is defined by

$$\mathbf{S}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}}.$$

Theorem

$\mathbf{X} = \{X_{i,j}\} \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ is a pure element in the regular polyball $\mathbf{B_n}(\mathcal{H})^-$, i.e. $\text{WOT-}\lim_{q_i \to \infty} \Phi_{X_i}^{q_i}(I) = 0$, if and only if there is a subspace $\mathcal{M} \subset F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{K}$ invariant under each $\mathbf{S}_{i,j} \otimes I$ such that

$$X_{i,j}^* = (\mathbf{S}_{i,j}^* \otimes I)|_{\mathcal{M}^\perp}.$$

Universal models

- The k-tuple $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$, where $\mathbf{S}_i := (\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i})$, is a pure element in the regular polyball $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$ and plays the role of *universal model* for the abstract regular polyball

$$\mathbf{B}_\mathbf{n}^- := \{\mathbf{B_n}(\mathcal{H})^- : \ \mathcal{H} \text{ is a Hilbert space}\}.$$

- Let $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-$ with $X_i := (X_{i,1}, \ldots, X_{i,n_i})$.
- Set $X_{i,\alpha_i} := X_{i,j_1} \cdots X_{i,j_p}$ if $\alpha_i = g^i_{j_1} \cdots g^i_{j_p} \in \mathbb{F}^+_{n_i}$ and $X_{i,g^i_0} := I$.

Noncommutative Berezin kernels

- If $\mathbf{X} = \{X_{i,j}\} \in \mathbf{B_n}(\mathcal{H})^-$, define the noncommutative Berezin kernel

$$\mathbf{K_X} : \mathcal{H} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\mathbf{\Delta_X}(I)(\mathcal{H})}$$

by setting

$$\mathbf{K_X} h := \sum_{\beta_i \in \mathbb{F}_{n_i}^+} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \mathbf{\Delta_X}(I)^{1/2} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^* h,$$

where the defect operator is given by

$$\mathbf{\Delta_X}(I) := (id - \Phi_{X_1}) \circ \cdots \circ (id - \Phi_{X_k})(I),$$

Noncommutative Berezin kernels

Theorem

The noncommutative Berezin kernel has the following properties :

(i) $\mathbf{K_X}$ *is a contraction and*

$$\mathbf{K_X^*K_X} = \lim_{s_k\to\infty} \ldots \lim_{s_1\to\infty} (id - \Phi_{X_k}^{s_k}) \circ \cdots \circ (id - \Phi_{X_1}^{s_1})(I),$$

where the limits are in the WOT.

(ii) *For any* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, n_i\}$,

$$\mathbf{K_X} X_{i,j}^* = (\mathbf{S}_{i,j}^* \otimes I)\mathbf{K_X}.$$

(iii) $\mathbf{X}$ *is pure if and only if* $\mathbf{K_X^*K_X} = I_{\mathcal{H}}$.

Noncommutative Berezin transforms

- The *Berezin transform* at $X \in \mathbf{B_n}(\mathcal{H})$ is the map $\mathcal{B}_{\mathbf{X}} : B(\otimes_{i=1}^k F^2(H_{n_i})) \to B(\mathcal{H})$ defined by

$$\mathcal{B}_{\mathbf{X}}[g] := \mathbf{K}_{\mathbf{X}}^*(g \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{X}}, \qquad g \in B(\otimes_{i=1}^k F^2(H_{n_i})).$$

- If $g \in C^*(\mathbf{S})$, the Berezin transform at $X \in \mathbf{B_n}(\mathcal{H})^-$, is defined by

$$\mathcal{B}_{\mathbf{X}}[g] := \lim_{r\to 1} \mathbf{K}_{r\mathbf{X}}^*(g \otimes I_{\mathcal{H}})\mathbf{K}_{r\mathbf{X}}, \qquad g \in C^*(\mathbf{S}),$$

and it is a unital completely positive linear map such that

$$\mathcal{B}_{\mathbf{X}}(\mathbf{S}_{(\alpha)}\mathbf{S}_{(\beta)}^*) = \mathbf{X}_{(\alpha)}\mathbf{X}_{(\beta)}^*,$$

where $\mathbf{S}_{(\alpha)} := \mathbf{S}_{1,\alpha_1} \cdots \mathbf{S}_{k,\alpha_k}$ if $(\alpha) := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{F}_{n_i}^+$.

Noncommutative polydomains : $\mathbf{D_q^m}$

Definition

Regular polydomain in $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$:

$$\mathbf{D_q^m}(\mathcal{H})^- := \left\{ \mathbf{X} : \ \Delta^{\mathbf{p}}_{\mathbf{q},\mathbf{X}}(I) \geq 0 \ \text{ for } \ \mathbf{0} \leq \mathbf{p} \leq \mathbf{m} \right\},$$

where $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{X}} := \left(id - \Phi_{q_1,X_1}\right)^{m_1} \circ \cdots \circ \left(id - \Phi_{q_k,X_k}\right)^{m_k}$ and

$$\Phi_{q_i,X}(Y) := \sum_\alpha a_{i,\alpha} X_{i,\alpha} Y X^*_{i,\alpha}, \quad q_i \in \mathbb{C}[Z_{i,1}, \ldots, Z_{i,n_i}],$$

with $a_{i,\alpha} \geq 0$, $a_{i,g_0^i} = 0$, $a_{i,g_j^i} \neq 0$ for $j = 1, \ldots, n_j$.

Noncommutative polydomains and varieties

- Noncommutative Berezin transforms
- Universal operator models and joint Beurling type invariant subspaces
- Dilation theory in polydomains (and varieties)
- Characteristic functions and operator models
- Free holomorphic functions in polydomains
- Similarity problems in noncommutative polydomains

- G. Popescu, Berezin transforms on noncommutative polydomains, Trans. Amer. Math. Soc (2015).
- G. Popescu, Berezin transforms on noncommutative varieties in polydomains, J. Funct. Anal. 265 (2013), 2500–2552.
- G. Popescu, Similarity problems in noncommutative polydomains, J. Funct. Anal. 267 (2014), 4446–4498.

Particular case : regular polydisc

- regular polydisc : $n_i = m_i = 1$ and $q_i := Z_i$.

- The regular polydisc case was considered by Brehmer, Bercovici, Douglas, Foias, Curto-Vasilescu, Timotin, Misra, Sarkar, Wick, and many others.

Particular case : Regular polyball

- regular polyball : $m_i = 1$, $q_i := Z_{i,1} + \cdots Z_{i,n_i}$.

Remark

The polyball case was recently considered by P. in connection with curvature invariant and Euler characteristic.

- G. Popescu, Curvature Invariant on Noncommutative Polyballs, *Adv. Math.*, in press.
- G. Popescu, Euler Characteristic on Noncommutative Polyballs, *J. Reine Angew. Math.*, in press.

Noncommutative regular polyballs

- If $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_k)$ and $\mathbf{z}_i = (z_{i,1}, \ldots, z_{i,n_i}) \in \mathbb{C}^{n_i}$; $\mathbf{X} := (\mathbf{X}_1, \ldots, \mathbf{X}_k)$ and $\mathbf{X}_i = (X_{i,1}, \ldots, X_{i,n_i}) \in B(\mathcal{H})^{n_i}$, we denote

$$\mathbf{z}\mathbf{X} := (\mathbf{z}_1\mathbf{X}_1, \ldots, \mathbf{z}_k\mathbf{X}_k) \text{ and } \mathbf{z}_i\mathbf{X}_i := (z_{i,1}X_{i,1}, \ldots, z_{i,k}X_{i,n_i}).$$

- If $\mathbf{r} := (r_1, \ldots, r_k)$, $r_i > 0$, we set $\mathbf{r}\mathbf{X} := (r_1\mathbf{X}_1, \ldots, r_k\mathbf{X}_k)$.

Definition

Let $G \subset B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$.

(i) G is a *complete Reinhardt set* if $\mathbf{z}\mathbf{X} \in G$ for any $\mathbf{X} \in G$ and $\mathbf{z} \in \overline{\mathbb{D}}^{n_1+\cdots+n_k}$.

(ii) If $\{(\log \|X_1\|, \ldots, \log \|X_k\|) : (X_1, \ldots, X_k) \in G, X_i \neq 0\}$ is a convex subset of $\mathbb{R}^k$, we say that G is *logarithmically convex*.

Noncommutative regular polyballs

Proposition

The regular polyball $\mathbf{B_n}(\mathcal{H})$ is a logarithmically convex complete Reinhardt domain such that

$$\mathbf{B_n}(\mathcal{H}) = \bigcup_{\mathbf{z}\in\overline{\mathbb{D}}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H}) = \bigcup_{\mathbf{z}\in\mathbb{D}^{n_1+\cdots+n_k}} \mathbf{z}\mathbf{B_n}(\mathcal{H})^-$$

and

$$\mathbf{B_n}(\mathcal{H}) = \bigcup_{0\leq r<1} r\mathbf{B_n}(\mathcal{H}) = \bigcup_{0\leq r<1} r\mathbf{B_n}(\mathcal{H})^-.$$

Power series and Abel type theorem

Theorem

If $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ *is a formal power series and* $\mathbf{r} = (r_1, \ldots, r_k)$, $r_i > 0$, *then the following statements hold.*

(i) *If the set*

$$A := \{\| r_1^{2p_1} \cdots r_k^{2p_k} \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i} A_{(\alpha)}^* A_{(\alpha)} \| : (p_1, \ldots, p_k) \in \mathbb{Z}_+^k\}$$

is bounded, then $\sum_{(p_1, \ldots, p_k) \in \mathbb{Z}_+^k} \| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = p_i} A_{(\alpha)} \otimes X_{(\alpha)} \|$ *is convergent in* $\mathbf{rB_n}(\mathcal{H})$ *and uniformly convergent on* $\mathbf{sB_n}(\mathcal{H})^-$ *for any* $\mathbf{s} = (s_1, \ldots, s_k)$ *with* $0 \le s_i < r_i$.

Abel type theorem

(ii) If the set A is unbounded, then the series

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \Big\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)} \otimes X_{(\alpha)} \Big\|$$

and

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)} \otimes X_{(\alpha)}$$

are divergent for some $\mathbf{X} \in \mathbf{rB_n}(\mathcal{H})^-$ and some Hilbert space $\mathcal{H}$.

Free holomorphic functions

Definition

A power series $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ is called *free holomorphic function* (with coefficients in $B(\mathcal{K})$) on the *abstract polyball* $\boldsymbol{\rho}\mathbf{B_n}$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_k)$, $\rho_i > 0$, if the series

$$\varphi(\mathbf{X}) := \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)} \otimes X_{(\alpha)}$$

is convergent in the operator norm topology for any $\mathbf{X} = \{X_{i,j}\}$ in $\boldsymbol{\rho}\mathbf{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$.

- $Hol(\boldsymbol{\rho}\mathbf{B_n})$ = all free holomorphic functions on $\boldsymbol{\rho}\mathbf{B_n}$ with scalar coefficients.

Cauchy type inequalities

Theorem

Let $F : \rho\mathbf{B_n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{min} B(\mathcal{H})$ be a free holomorphic function with representation

$$F(\mathbf{X}) = \sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)} \otimes X_{(\alpha)}.$$

Let $\mathbf{r} = (r_1, \ldots, r_k)$ be such that $0 < r_i < \rho_i$ and define $M(\mathbf{r}) := \sup_{\mathbf{X}\in \mathbf{r}\mathbf{B_n}(\mathcal{H})^-} \|F(\mathbf{X})\|$. Then, for each $(p_1, \ldots, p_k) \in \mathbb{Z}_+^k$,

$$\|\sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A^*_{(\alpha)} A_{(\alpha)}\|^{1/2} \leq \frac{1}{r_1^{p_1}\cdots r_k^{p_k}} M(\mathbf{r})$$

and $M(\mathbf{r}) = \|F(\mathbf{rS})\|$, where $\mathbf{S}$ is the universal model of $\mathbf{B_n}$.

Liouville type result

- F is an *entire function* in $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ if it is free holomorphic on every regular polyball $\rho\mathbf{B_n}(\mathcal{H})$, $\rho > 0$.

Corollary

If $F : B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} \to B(\mathcal{K}) \otimes_{min} B(\mathcal{H})$ *is an entire function with the property that there is a constant* $C > 0$ *and* $(p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ *such that*

$$\|F(\mathbf{X})\| \leq C \| \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = q_i} \mathbf{X}_{(\alpha)} \mathbf{X}_{(\alpha)}^* \|^{1/2}$$

for any $\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, *then* F *is a polynomial of degree at most* $q_1 + \cdots + q_k$. *In particular, a bounded free holomorphic function must be constant.*

Hadamard type formula

Theorem

Let $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ *be a formal power series and define* $\gamma \in [0, \infty]$ *by setting*

$$\frac{1}{\gamma} := \limsup_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \Big\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)}^* A_{(\alpha)} \Big\|^{\frac{1}{2(p_1+\cdots+p_k)}}.$$

Then the following statements hold.

(i) *The series*

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \Big\| \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)} \otimes X_{(\alpha)} \Big\|, \qquad \mathbf{X} \in \gamma \mathbf{B_n}(\mathcal{H}),\ \textit{is}$$

convergent. Moreover, the convergence is uniform on $r\mathbf{B_n}(\mathcal{H})^-$ *if* $0 \le r < \gamma$.

Hadamard type formula

(ii) For any $s > \gamma$, there is a Hilbert space $\mathcal{H}$ and $\mathbf{Y} \in s\mathbf{B_n}(\mathcal{H})^-$ such that the series

$$\sum_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \sum_{\alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=p_i} A_{(\alpha)} \otimes Y_{(\alpha)}$$

is divergent in the operator norm topology.

Free holomorphic functions

Corollary

A formal power series $\varphi = \sum_{(\alpha)} A_{(\alpha)} \otimes Z_{(\alpha)}$ is a free holomorphic function on the abstract polyball $\boldsymbol{\rho}\mathbf{B_n}$, where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_k)$, $\rho_i > 0$, if and only if the series

$$\sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+ \times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} A_{(\alpha)} \otimes r^q \rho_1^{|\alpha_1|} \cdots \rho_k^{|\alpha_k|} \mathbf{S}_{(\alpha)}$$

is convergent in the operator norm topology for any $r \in [0, 1)$.

- The set $Hol(\boldsymbol{\rho}\mathbf{B_n})$ of all free holomorphic functions (with scalar coefficients) on $\boldsymbol{\rho}\mathbf{B_n}$ is an algebra.

Bounded free holomorphic functions

- $H^\infty(\mathbf{B_n})$ denotes the set of all $\varphi \in Hol(\mathbf{B_n})$ such that

$$\|\varphi\|_\infty := \sup_{\mathbf{X},\mathcal{H}} \|\varphi(\mathbf{X})\| < \infty.$$

- $H^\infty(\mathbf{B_n})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$.
- For each $p \in \mathbb{N}$, we define $\|\cdot\|_p : M_{p\times p}(H^\infty(\mathbf{B_n})) \to [0,\infty)$ by setting

$$\|[\varphi_{st}]_{p\times p}\|_p := \sup_{\mathbf{X},\mathcal{H}} \|[\varphi_{st}(\mathbf{X})]_{p\times p}\|.$$

The norms $\|\cdot\|_p$, $p \in \mathbb{N}$, determine an operator algebra structure on $H^\infty(\mathbf{B_n})$, in the sense of Ruan.

Free holomorphic functions

Theorem

Let $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and $\mathbf{m} = (m_1, \dots, m_q) \in \mathbb{N}^q$. If $G : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_m}(\mathcal{H})$ and $F : \mathbf{B_m}(\mathcal{H}) \to B(\mathcal{H})\bar{\otimes}_{min} B(\mathcal{E}, \mathcal{G})$ are free holomorphic functions on regular polyballs, then $F \circ G$ is a free holomorphic function on $\mathbf{B_n}(\mathcal{H})$.

- $\mathbf{B_n}(\mathcal{H})$ is a complete Reinhardt domain and

$$B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} = \bigcup_{\rho > 0} \rho \mathbf{B_n}(\mathcal{H}).$$

- The Minkovski functional associated with $\mathbf{B_n}(\mathcal{H})$ is the function $m_{\mathbf{B_n}} : B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} \to [0, \infty)$ given by

$$m_{\mathbf{B_n}}(\mathbf{X}) := \inf \{ r > 0 : \ \mathbf{X} \in r\mathbf{B_n}(\mathcal{H}) \}.$$

Minkovski functional

Proposition

The Minkovski functional associated with $\mathbf{B_n}(\mathcal{H})$ has the following properties :

(i) $m_{\mathbf{B_n}}(\lambda \mathbf{X}) = |\lambda| m_{\mathbf{B}}(\mathbf{X})$ *for* $\lambda \in \mathbb{C}$ *;*

(ii) $m_{\mathbf{B_n}}$ *is upper semicontinuous ;*

(iii) $\mathbf{B_n}(\mathcal{H}) = \{\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} : \ m_{\mathbf{B_n}}(\mathbf{X}) < 1\}$ *;*

(iv) $\mathbf{B_n}(\mathcal{H})^- = \{\mathbf{X} \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} : \ m_{\mathbf{B_n}}(\mathbf{X}) \leq 1\}$ *;*

(v) *There is a polyball* $r\mathbf{P_n}(\mathcal{H}) \subset \mathbf{B_n}(\mathcal{H})$ *for some* $r \in (0, 1)$*, where* $m_{\mathbf{B_n}}$ *is continuous.*

Free partial derivation

Let $\mathbb{C}\langle Z_{i,j}\rangle$ be the algebra of all polynomials in indeterminates $Z_{i,j}$. The free partial derivation $\frac{\partial}{\partial Z_{i,j}}$ on $\mathbb{C}\langle Z_{i,j}\rangle$ is defined as the unique linear operator, satisfying the conditions

$$\frac{\partial I}{\partial Z_{i,j}} = 0, \quad \frac{\partial Z_{i,j}}{\partial Z_{i,j}} = I, \quad \frac{\partial Z_{i,j}}{\partial Z_{s,q}} = 0 \text{ if } (i,j) \neq (s,q)$$

and

$$\frac{\partial (fg)}{\partial Z_{i,j}} = \frac{\partial f}{\partial Z_{i,j}} g + f \frac{\partial g}{\partial Z_{i,j}}$$

for any $f, g \in \mathbb{C}\langle Z_{i,j}\rangle$. The same definition extends to formal power series in the noncommuting indeterminates $Z_{i,j}$.

Schwarz type result

Theorem

Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^p$ be a bounded free holomorphic function with $\|F\|_\infty \leq 1$. If $F(0) = 0$, then

$$\|F(\mathbf{X})\| \leq m_{\mathbf{B_n}}(\mathbf{X}) < 1 \quad \text{and} \quad m_{\mathbf{B_n}}(\mathbf{X}) \leq \|\mathbf{X}\|, \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

In particular, if $p = 1$, the free holomorphic function

$$\psi(\mathbf{X}) = \sum_{i=1}^{k} \sum_{j=1}^{n_j} \frac{\partial F}{\partial Z_{i,j}}(0) X_{i,j}, \qquad \mathbf{X} = (X_{i,j}) \in \mathbf{B_n}(\mathcal{H}),$$

has the property that $\|\psi(\mathbf{X})\| \leq m_{\mathbf{B_n}}(\mathbf{X}) < 1$.

Maximum principle

Theorem

Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$ be a bounded free holomorphic function. If there exists $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that

$$\|F(\mathbf{X})\| \leq \|F(\mathbf{X}_0)\|, \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

then F must be a constant.

Theorem

Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})^p$ be a bounded free holomorphic function with $\|F(0)\| < \|F\|_\infty$. Then there is no $\mathbf{X}_0 \in \mathbf{B_n}(\mathcal{H})$ such that $\|F(\mathbf{X}_0)\| = \|F\|_\infty$.

Cartan type results

- Consider the set

$$\Lambda_{\mathbf{n}} := \{(i,j) : \ i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}\}$$

  with the lexicographic order and let

$$F(\mathbf{X}) = [F_{s,t}(\mathbf{X}) : \ (s,t) \in \Lambda_{\mathbf{n}}]$$

  be a free holomorphic function on $\mathbf{B_n}$.

- Define $F'(0)$ as the linear operator on $\mathbb{C}^{n_1+\cdots+n_k}$ having the matrix $\left[\frac{\partial F_{i,j}}{\partial Z_{s,t}}(0)\right]_{\Lambda_{\mathbf{n}}\times\Lambda_{\mathbf{n}}}$

Cartan type results

Theorem

Let $F : \mathbf{B_n} \to \mathbf{B_n}$ *be a free holomorphic function such that* $F(0) = 0$ *and* $F'(0) = I$. *Then*

$$F(\mathbf{X}) = \mathbf{X}, \qquad \mathbf{X} \in \mathbf{B_n}.$$

- A map $F : \mathbf{B_n} \to \mathbf{B_n}$ is called free biholomorphic if F is free homolorphic, one-to-one and onto, and has free holomorphic inverse.
- The automorphism group of $\mathbf{B_n}$ is denoted by $Aut(\mathbf{B_n})$.

Cartan type results

Theorem

Let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ *and let* $F : \mathbf{B_n} \to \mathbf{B_n}$ *be a free biholomorphic function with* $F(0) = 0$*. Then there are unitary operators* $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$*, and a permutation* $\sigma \in \mathcal{S}_k$ *with the property that* $n_{\sigma^{-1}(i)} = n_i$ *for* $i \in \{1, \ldots, k\}$ *such that*

$$F = p_\sigma \circ \mathbf{\Phi_U},$$

where p_σ *and* $\mathbf{\Phi_U}$ *are free holomorphic functions on* $\mathbf{B_n}$ *defined by* $p_\sigma(\mathbf{X}) = (X_{\sigma(1)}, \ldots, X_{\sigma(k)})$ *and*

$$\mathbf{\Phi_U}(\mathbf{X}) := \mathbf{XU} := [X_1 U_1, \ldots, X_k U_k],$$

where $\mathbf{U} := U_1 \oplus \cdots \oplus U_k$*. Moreover, the converse is also true.*

Cartan type results

Theorem

Let $F : \mathbf{B_n} \to \mathbf{B_n}$ be a free holomorphic function such that $F'(0)$ is a unitary operator on $\mathbb{C}^{n_1+\cdots+n_k}$. Then F is a free holomorphic automorphism of $\mathbf{B_n}$ and

$$F(\mathbf{X}) = \mathbf{X}[F'(0)]^t, \qquad \mathbf{X} \in \mathbf{B_n},$$

where $^\tau$ denotes the transpose.

Structure of automorphisms

Theorem

Let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ *and let* $\boldsymbol{\Psi} \in \mathrm{Aut}(\mathbf{B_n})$. *If* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$, *then there are unique unitary operators* $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, *and a unique permutation* $\sigma \in \mathcal{S}_k$ *with* $n_{\sigma(i)} = n_i$ *such that*

$$\boldsymbol{\Psi} = p_\sigma \circ \boldsymbol{\Phi}_{\mathbf{U}} \circ \boldsymbol{\Psi}_{\boldsymbol{\lambda}},$$

where $\mathbf{U} := U_1 \oplus \cdots \oplus U_k$, $\boldsymbol{\Psi}_{\boldsymbol{\lambda}} := (\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_k})$, *and*

$$\Psi_{\lambda_i}(Y_i) = -\Theta_{\lambda_i}(Y_i) := \lambda_i - \Delta_{\lambda_i}\left(I_{\mathcal{K}} - Y_i\lambda_i^*\right)^{-1} Y_i \Delta_{\lambda_i^*},$$

where Θ_{λ_i} *is the characteristic function of the row contraction* λ_i, *and* $\Delta_{\lambda_i} = (1 - \|\lambda_i\|_2^2)^{1/2}$, $\Delta_{\lambda_i^*} = (I_{\mathbb{C}^{n_i}} - \lambda_i^*\lambda_i)^{1/2}$ *are the defect operators of* λ_i.

Biholomorphic polyballs

Theorem

Let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ *and* $\mathbf{m} = (m_1, \ldots, m_q) \in \mathbb{N}^q$*. Then*

$$Bih(\mathbf{B_n}, \mathbf{B_m}) \neq \emptyset$$

if and only if $k = q$ *and there is a permutation* $\sigma \in \mathcal{S}_k$ *such that* $m_{\sigma(i)} = n_i$ *for any* $i \in \{1, \ldots, k\}$*. Moreover, any free biholomorphic function* $F : \mathbf{B_n} \to \mathbf{B_m}$ *is up to a permutation of* $(m_1, \ldots, m_k)$ *an automorphism of the noncommutative regular polyball* $\mathbf{B_n}$*.*

Properties of automorphisms

Theorem

Let $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n})$, *where* $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, *and let* $\hat{\mathbf{\Psi}} = (\hat{\Psi}_1, \ldots, \hat{\Psi}_k)$ *be the boundary function with respect to the universal model* $\mathbf{S} = \{\mathbf{S}_{i,j}\}$*. The following statements hold.*

(i) $\mathbf{\Psi}$ *is a free holomorphic function on the regular polyball* $\gamma\mathbf{B_n}$ *for some* $\gamma > 1$.

(ii) *The boundary function* $\hat{\mathbf{\Psi}}$ *with respect to* $\mathbf{S}$ *is a pure element in the polyball* $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$ *and* $\hat{\mathbf{\Psi}} := \lim_{r\to 1} \mathbf{\Psi}(r\mathbf{S}) = \mathbf{\Psi}(\mathbf{S})$*. Each* $\hat{\Psi}_i = (\hat{\Psi}_{i,1}, \ldots, \hat{\Psi}_{i,n_i})$ *is an isometry with entries in the noncommutative disk algebra generated by* $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$ *and the identity.*

(iii) $\mathbf{\Psi}$ *is a homeomorphism of* $\mathbf{B_n}(\mathcal{H})^-$ *onto* $\mathbf{B_n}(\mathcal{H})^-$.

Properties of automorphisms

(iv) If $\Psi \in Aut(\mathbf{B_n})$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) = \boldsymbol{\Psi}^{-1}(0)$, then the identity

$$\boldsymbol{\Delta}_{\boldsymbol{\Psi}(\mathbf{X})}(I) = \boldsymbol{\Delta}_{\boldsymbol{\lambda}} \left[\prod_{i=1}^{k} (I_{\mathcal{H}} - X_i\lambda_i^*)^{-1}\right] \boldsymbol{\Delta}_X(I) \left[\prod_{i=1}^{k} (I_{\mathcal{H}} - \lambda_i X_i^*)^{-1}\right]$$

holds for any $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})^-$, where $\boldsymbol{\Delta}_{\boldsymbol{\lambda}} = \prod_{i=1}^{k}(1 - \|\lambda_i\|_2^2)$.

(v) $\operatorname{rank} \boldsymbol{\Delta}_{\hat{\Psi}}(I) = 1$ and $\hat{\boldsymbol{\Psi}}$ is unitarily equivalent to the universal model $\mathbf{S}$.

$Aut(\mathbf{B_n}) \simeq \mathrm{Aut}((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1)$

Theorem

The map $\Lambda : Aut(\mathbf{B_n}) \to \mathrm{Aut}((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1)$ *defined by*

$$\Lambda(\boldsymbol{\Psi})(\mathbf{z}) := (\mathbf{B_z}[\hat{\boldsymbol{\Psi}}_1], \ldots, \mathbf{B_z}[\hat{\boldsymbol{\Psi}}_k]) \qquad \mathbf{z} \in (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1,$$

is a group isomorphism, where $\hat{\boldsymbol{\Psi}}$ *is the boundary function of* $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_k) \in Aut(\mathbf{B_n})$ *with respect to the universal model* $\mathbf{S}$ *and* $\mathbf{B_z}$ *is the noncommutative Berezin transform at* $\mathbf{z}$*.*

Automorphisms of Cuntz-Toeplitz algebras

Proposition

A free holomorphic function $F : \mathbf{B_n}(\mathcal{H}) \to \mathbf{B_n}(\mathcal{H})^-$ has a continuous extension (also denoted by F) to $\mathbf{B_n}(\mathcal{H})^-$ if and only if the boundary function $\hat{F}$ has the entries in the noncommutative polyball algebra $\mathcal{A}_\mathbf{n}$ and $\hat{F} \in \mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$. Moreover,

$$\mathcal{B}_{F(\mathbf{X})}[g] = \mathcal{B}_\mathbf{X}[\mathcal{B}_{\hat{F}}[g]]$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ and $g \in C^*(\mathbf{S})$. If, in addition, $\hat{F}$ is a pure element of the polyball $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$, then the same relation holds for any pure element $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ and $g \in \mathbf{F_n^\infty}$.

Automorphisms of Cuntz-Toeplitz algebras

Theorem

Any automorphism Γ of the Cuntz-Toeplitz C^*-algebra $C^*(\mathbf{S})$ which leaves invariant the noncommutative polyball algebra $\mathcal{A}_\mathbf{n}$, i.e. $\Gamma(\mathcal{A}_\mathbf{n}) = \mathcal{A}_\mathbf{n}$, has the form

$$\Gamma(g) := \mathcal{B}_{\hat{\Psi}}[g] = \mathbf{K}^*_{\hat{\Psi}}[g \otimes I_{\mathcal{D}_{\hat{\Psi}}}]\mathbf{K}_{\hat{\Psi}}, \qquad g \in C^*(\mathbf{S}),$$

where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$ and $\mathcal{B}_{\hat{\Psi}}$ is the noncommutative Berezin transform at the boundary function $\hat{\Psi}$. In this case, the noncommutative Berezin kernel $\mathbf{K}_{\hat{\Psi}}$ is a unitary operator and Γ is a unitary implemented automorphism of $C^*(\mathbf{S})$. Moreover, we have

$$\mathrm{Aut}_{\mathcal{A}_\mathbf{n}}(C^*(\mathbf{S})) \simeq \mathrm{Aut}(\mathbf{B_n}).$$

Automorphisms of Cuntz-Toeplitz algebras

- When $k = 1$, we recover a theorem (P., Crelle 2010) which strengthened Voiculescu's result (Lect. Notes Math., 1985) on automorphisms of the Cuntz-Toeplitz algebra $C^*(S_1, \ldots, S_n)$.

Automorphisms of Cuntz-Toeplitz algebras

- The Cuntz-Toeplitz algebra $\mathcal{T}_n$ is the unique unital C^*-algebra generated by n isometries $s_1, \ldots, s_n$ satisfying relations $s_i^* s_j = \delta_{ij} 1$ and $s_1 s_1^* + \cdots + s_n s_n^* < 1$.
- The noncommutative disc algebra $\mathcal{A}_n$ is the unique non-self-adjoint closed algebra generated $s_1, \ldots, s_n$ and the identity.
- The Cuntz algebra $\mathcal{O}_n$ is uniquely defined as the C^*-algebra generated by $n \geq 2$ isometries satisfying relations $\sigma_i^* \sigma_j = \delta_{ij} 1$ and $\sigma_1 \sigma_1^* + \cdots + \sigma_n \sigma_n^* = 1$
- The C^*-algebra $C^*(\mathbf{S})$ generated by the universal model $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ is $*$-isomorphic to $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$.
- $\mathcal{A}_{n_i}$ can be seen as a subalgebra of $\mathcal{T}_{n_i}$ and $\boldsymbol{\mathcal{A}}_{\mathbf{n}} \simeq \mathcal{A}_{n_1} \otimes_{min} \cdots \otimes_{min} \mathcal{A}_{n_k}$.

Automorphisms of Cuntz algebras

- Using the short exact sequence obtained by Cuntz, one can deduce that there is a a surjective $*$-representation $\chi : C^*(\mathbf{S}) \to \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ such that $\chi(\mathbf{S}_{i,j}) = \boldsymbol{\sigma}_{i,j}$, where

$$\boldsymbol{\sigma}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes \sigma_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

  where $\{\sigma_{i,j}\}_{j=1}^{n_i}$ is a set of generators of the Cuntz algebra $\mathcal{O}_{n_i}$.
- The closed non-seladjoint algebra $Alg(1, \sigma_i)$ generated by $\{\sigma_{i,j}\}_{j=1}^{n_i}$ and the identity is completely isometric isomorphic to the noncommutative disc algebra $\mathcal{A}_{n_i}$. Consequently, one can see $\boldsymbol{\mathcal{A}}_{\mathbf{n}} \simeq \mathcal{A}_{n_1} \otimes_{min} \cdots \otimes_{min} \mathcal{A}_{n_k}$ as a subalgebra of $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$.

Automorphisms of Cuntz algebras

Corollary

Let $\mathbf{n} = (n_1, \ldots n_k) \in \mathbb{N}^k$. Each holomorphic automorphism of the regular polyball $\mathbf{B_n}$ induces an automorphism of the C^*-algebra $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ which leaves invariant the non-self-adjoint subalgebra $\mathcal{A}_{n_1} \otimes_{min} \cdots \otimes_{min} \mathcal{A}_{n_k}$.

Automorphisms of the polyball algebra

Theorem

Any unitarily implemented automorphism of the noncommutative polyball algebra $\mathcal{A}_\mathbf{n}$ is the Berezin transform $\mathcal{B}_{\hat{\Psi}}|_{\mathcal{A}_\mathbf{n}}$ of a boundary function $\hat{\Psi}$, where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$. Moreover, we have

$$\mathrm{Aut}_u(\mathcal{A}_\mathbf{n}) \simeq \mathrm{Aut}(\mathbf{B_n}).$$

Automorphisms of the polyball algebra

- $A(\mathbf{B_n})$ is the set of all elements g in $Hol(\mathbf{B_n})$ such that the mapping
$$\mathbf{B_n}(\mathcal{H}) \ni \mathbf{X} \mapsto g(\mathbf{X}) \in B(\mathcal{H})$$
has a continuous extension to $[\mathbf{B_n}(\mathcal{H})]^-$ for any $\mathcal{H}$.
- $A(\mathbf{B_n})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$, and it has an operator algebra structure under the norms $\|\cdot\|_p$, $p \in \mathbb{N}$.
- The map $\Phi : A(\mathbf{B_n}) \to \mathcal{A}_\mathbf{n}$ defined by
$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\right) := \sum_{(\alpha)} a_{(\alpha)} \mathbf{S}_{(\alpha)}$$
is a completely isometric isomorphism of operator algebras.

Automorphisms of the polyball algebra

- We have $\Phi(g) = \hat{g} := \lim_{r\to 1} g(r\mathbf{S})$ in the operator norm topology, and and $\Phi^{-1}(\varphi) = \boldsymbol{\mathcal{B}}[\varphi]$ for $\varphi \in \boldsymbol{\mathcal{A}}_\mathbf{n}$, where

$$\boldsymbol{\mathcal{B}}[\varphi] : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$$

is defined by

$$\boldsymbol{\mathcal{B}}[\varphi](\mathbf{X}) := \boldsymbol{\mathcal{B}}_\mathbf{X}[\varphi], \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

- If $\Lambda : A(\mathbf{B_n}) \to A(\mathbf{B_n})$ is an algebraic homomorphism, it induces a unique homomorphism $\tilde{\Lambda} : \boldsymbol{\mathcal{A}}_n \to \boldsymbol{\mathcal{A}}_n$ such that $\Lambda\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}\tilde{\Lambda}$. The homomorphisms Λ and $\tilde{\Lambda}$ uniquely determine each other by the formulas :

$$(\Lambda f)(\mathbf{X}) = \boldsymbol{\mathcal{B}}_\mathbf{X}[\tilde{\Lambda}(\hat{f})], \qquad f \in A(\mathbf{B_n}),\ \mathbf{X} \in \mathbf{B_n}(\mathcal{H}), \quad \text{and}$$

$$\tilde{\Lambda}(\hat{f}) = \widehat{\Lambda(f)}, \qquad \hat{f} \in \boldsymbol{\mathcal{A}}_\mathbf{n}.$$

Automorphisms of the polyball algebra

Theorem

Let $\Lambda : A(\mathbf{B_n}) \to A(\mathbf{B_n})$ be a unital algebraic automorphism. Then the following statements are equivalent.

(i) *$\tilde{\Lambda}$ is a unitarily implemented automorphism of $\mathcal{A}_\mathbf{n}$.*

(ii) *There is $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ such that*

$$\Lambda(f) = f \circ \varphi, \qquad f \in A(\mathbf{B_n}).$$

(iii) *$\tilde{\Lambda}$ is continuous and $\{\tilde{\Lambda}(\mathbf{S}_{i,j})\}$ and $\{\tilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ are in the polyball $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$, where $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ is the universal model of the regular polyball $\mathbf{B_n}$.*

Automorphisms of the polyball algebra

Remark

If $\Lambda : A(\mathbf{B_n}) \to A(\mathbf{B_n})$ is a unital algebraic homomorphism and at least one of $n_1, \ldots, n_k$ is ≥ 2, then $\tilde{\Lambda}$ is automatically continuous.

Automorphisms of the algebra $\mathbf{F}_\mathbf{n}^\infty$

Theorem

Any unitarily implemented automorphism of the noncommutative Hardy algebra $\mathbf{F}_\mathbf{n}^\infty$ is the Berezin transform $\mathcal{B}_{\hat{\Psi}}$ of a boundary function $\hat{\Psi}$, where $\Psi \in \mathrm{Aut}(\mathbf{B_n})$. Moreover, we have

$$\mathrm{Aut}_u(\mathbf{F}_\mathbf{n}^\infty) \simeq \mathrm{Aut}(\mathbf{B_n}).$$

Remark

The case $k = 1$ was considered by Davidson and Pitts, and P.

Automorphisms of the Hardy algebra $H^\infty(\mathbf{B_n})$

- $H^\infty(\mathbf{B_n})$ is the set of all bounded elements in $Hol(\mathbf{B_n})$.
- $H^\infty(\mathbf{B_n})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$, and it has an operator algebra structure under the norms $\|\cdot\|_p$, $p \in \mathbb{N}$.
- The map $\Phi : H^\infty(\mathbf{B_n}) \to \mathbf{F_n^\infty}$ defined by

$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\right) := \sum_{(\alpha)} a_{(\alpha)} \mathbf{S}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras.

Automorphisms of the Hardy algebra $H^\infty(\mathbf{B_n})$

- We have $\Phi(g) = \hat{g} := \text{SOT-}\lim_{r\to 1} g(r\mathbf{S})$ in the strong operator topology, and $\Phi^{-1}(\varphi) = \mathcal{B}[\varphi]$ for $\varphi \in \mathbf{F_n^\infty}$, where

$$\mathcal{B}[\varphi] : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$$

is defined by

$$\mathcal{B}[\varphi](\mathbf{X}) := \mathcal{B}_\mathbf{X}[\varphi], \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

- If $\Lambda : H^\infty(\mathbf{B_n}) \to H^\infty(\mathbf{B_n})$ is an algebraic homomorphism, it induces a unique homomorphism $\tilde{\Lambda} : \mathbf{F_n^\infty} \to \mathbf{F_n^\infty}$ such that $\Lambda\mathcal{B} = \mathcal{B}\tilde{\Lambda}$. The homomorphisms Λ and $\tilde{\Lambda}$ uniquely determine each other by the formulas :

$$(\Lambda f)(\mathbf{X}) = \mathcal{B}_\mathbf{X}[\tilde{\Lambda}(\hat{f})], \qquad f \in H^\infty(\mathbf{B_n}),\ \mathbf{X} \in \mathbf{B_n}(\mathcal{H}), \quad \text{and}$$

$$\tilde{\Lambda}(\hat{f}) = \widehat{\Lambda(f)}, \qquad \hat{f} \in \mathbf{F_n^\infty}.$$

Automorphisms of the Hardy algebra $H^\infty(\mathbf{B_n})$

Theorem

Let $\Lambda : H^\infty(\mathbf{B_n}) \to H^\infty(\mathbf{B_n})$ *be a unital algebraic automorphism. Then the following statements are equivalent.*

(i) $\tilde{\Lambda}$ *is a unitarily implemented automorphism of* $\mathbf{F_n^\infty}$.

(ii) *There is* $\varphi \in \mathrm{Aut}(\mathbf{B_n})$ *such that*

$$\Lambda(f) = f \circ \varphi, \qquad f \in H^\infty(\mathbf{B_n}).$$

(iii) $\tilde{\Lambda}$ *is norm-continuous and WOT-continuous such that* $\{\tilde{\Lambda}(\mathbf{S}_{i,j})\}$ *and* $\{\tilde{\Lambda}^{-1}(\mathbf{S}_{i,j})\}$ *are in the polyball* $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$, *where* $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ *is the universal model of the regular polyball* $\mathbf{B_n}$.

Unitary projective representation for $Aut(\mathbf{B_n})$

- Each $\mathbf{\Psi} \in \mathrm{Aut}(\mathbf{B_n})$ has a unique representation $\mathbf{\Psi} = p_\sigma \circ \Phi_{\mathbf{U}} \circ \mathbf{\Psi}_{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} := \mathbf{\Phi}^{-1}(0)$ and $\mathbf{U} = U_1 \oplus \cdots \oplus U_k$ with $U_i \in \mathcal{U}(\mathbb{C}^{n_i})$, the unitary group on $\mathbb{C}^{n_i}$.
- This representation generates a homeomorphism

$$\chi : \mathrm{Aut}(\mathbf{B_n}) \to \Sigma \times \mathcal{U}(\mathbb{C}^{n_1}) \times \cdots \times \mathcal{U}(\mathbb{C}^{n_k}) \times \mathbf{P_n}(\mathbb{C}),$$

by setting $\chi(\mathbf{\Psi}) := (\sigma, U_1, \cdots, U_k, \boldsymbol{\lambda})$, where Σ is the discrete subgroup

$$\Sigma := \{\sigma \in \mathcal{S}_k : \ (n_{\sigma(1)}, \ldots, n_{\sigma(k)}) = (n_1, \ldots, n_k)\}.$$

Unitary projective representation for $Aut(\mathbf{B_n})$

Theorem

The free holomorphic automorphism group $\mathrm{Aut}(\mathbf{B_n})$ *is a* σ*-compact, locally compact topological group with respect to the topology induced by the metric* $d_{\mathbf{B_n}}$ *defined by*

$$d_{\mathbf{B_n}}(\phi,\psi) := \|\phi-\psi\|_\infty + \|\phi^{-1}(0)-\psi^{-1}(0)\|, \quad \phi,\psi \in \mathrm{Aut}(\mathbf{B_n}).$$

Corollary

Let $\mathbf{n}=(n_1,\ldots,n_k)\in\mathbb{N}^k$ *and*

$$\Sigma := \{\sigma \in \mathcal{S}_k :\ (n_{\sigma(1)},\ldots,n_{\sigma(k)}) = (n_1,\ldots,n_k)\}.$$

The free holomorphic automorphism group $\mathrm{Aut}(\mathbf{B_n})$ *has* $\mathrm{card}(\Sigma)$ *path connected components.*

Unitary projective representation for $Aut(\mathbf{B_n})$

Definition

A map $\pi : \mathrm{Aut}(\mathbf{B_n}) \to \mathcal{U}(\mathcal{K})$ is called (unitary) projective representation if the following conditions are satisfied :

(i) $\pi(id) = I$, where *id* is the identity on $\mathbf{B_n}$;

(ii) $\pi(\mathbf{\Phi})\pi(\mathbf{\Psi}) = c_{(\mathbf{\Phi},\mathbf{\Psi})}\pi(\mathbf{\Phi} \circ \mathbf{\Psi})$, for any $\mathbf{\Phi}, \mathbf{\Psi} \in \mathrm{Aut}(\mathbf{B_n})$, where $c_{(\mathbf{\Phi},\mathbf{\Psi})} \in \mathbb{T}$;

(iii) the map $\mathrm{Aut}(\mathbf{B_n}) \ni \mathbf{\Phi} \mapsto \langle \pi(\mathbf{\Phi})\xi, \eta \rangle \in \mathbb{C}$ is continuous for each $\xi, \eta \in \mathcal{K}$.

Unitary projective representation for $Aut(\mathbf{B_n})$

Theorem

For each $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_k) \in \mathrm{Aut}(\mathbf{B_n})$, $\Psi_i = (\Psi_{i,1}, \ldots, \Psi_{i,n_i})$, *there is a unitary operator* $U_{\boldsymbol{\Psi}} \in B(\otimes_{i=1}^k F^2(H_{n_i}))$ *satisfying the relations* $\Psi_{i,j}(\mathbf{S}) = U_{\boldsymbol{\Psi}}^* \mathbf{S}_{i,j} U_{\boldsymbol{\Psi}}$ *and*

$$U_{\boldsymbol{\Psi}} U_{\boldsymbol{\Phi}} = c_{(\boldsymbol{\Psi},\boldsymbol{\Phi})} U_{\boldsymbol{\Psi}\circ\boldsymbol{\Phi}}, \quad \boldsymbol{\Phi}, \Psi \in \mathrm{Aut}(\mathbf{B_n})$$

for some $c_{(\boldsymbol{\Phi},\boldsymbol{\Psi})} \in \mathbb{T}$. *Moreover, the map* $\boldsymbol{\Psi} \to U_{\boldsymbol{\Psi}}^*$ *is continuous from the uniform topology to the strong operator topology, and the map* $\pi : \mathrm{Aut}(\mathbf{B_n}) \to B(\otimes_{i=1}^k F^2(H_{n_i}))$ *defined by* $\pi(\boldsymbol{\Psi}) := U_{\boldsymbol{\Psi}}$ *is a projective representation of the automorphism group* $\mathrm{Aut}(\mathbf{B_n})$.

THANK YOU