Operators on Matrix Weighted L^2 Spaces

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Scalar Setting

Defn. 1: w is a *weight* if $w \in L^1_{loc}(\mathbb{R})$ and $w(x) > 0$ almost everywhere. Then $L^2(w) \equiv \left\{ f : \int |f|^2 w < \infty \right\}$.

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A weight w is an A_2 *weight* if

$$[w]_{A_2} \equiv \sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-1} \right) \equiv \sup_I \langle w \rangle_I \left\langle w^{-1} \right\rangle_I < \infty.$$

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Defn. 2: T is a *CZO* if $T \in \mathcal{B}(L^2(\mathbb{R}))$ and

$$Tf(x) = \int_{\mathbb{R}} K(x,y) f(y) dy \quad x \notin \mathrm{supp} f,$$

where $K(x,y)$ satisfies "boundedness" and "smoothness" properties away from the diagonal $\{(x,y) : x = y\}$.

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Ex. The Hilbert Transform: $Hf(x) = \dfrac{1}{\pi} \displaystyle\int_{\mathbb{R}} \frac{f(y)}{x-y} dx \qquad \text{if } x \notin \text{supp}(f).$

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If T is a CZO and w is an A_2 weight, then $\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$.

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Dyadic Shifts

The *dyadic grid* is: $\mathcal{D} \equiv \left\{2^j[k, k+1) : j, k \in \mathbb{Z}\right\}$. The *Haar basis* $\{h_I\}$ is the set of orthonormal functions $h_I \equiv |I|^{-\frac{1}{2}} \left(\mathbf{1}_{I_+} - \mathbf{1}_{I_-}\right)$ for all $I \in \mathcal{D}$.

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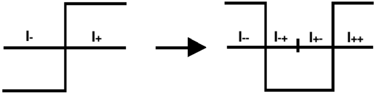
Defn. *A Dyadic Shift of Complexity r* is a CZO that sends Haar functions h_I to weighted combinations of other Haar functions h_J satisfying $d_{tree}(I, J) \leq r$.

Ex. One important dyadic shift is: $\mathbb{S} : h_I \mapsto \frac{1}{\sqrt{2}} \left(h_{I_+} - h_{I_-} \right)$



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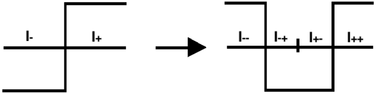

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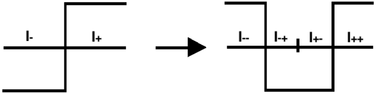

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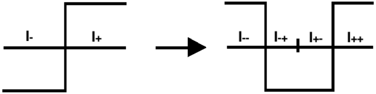

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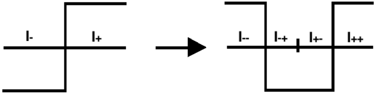

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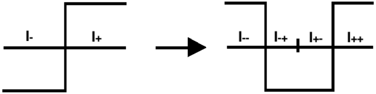

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T(1) Theorem (Nazarov-Treil-Volberg 2008)

Let w, v be weights and let T be a band operator of radius r. If for all $I \in \mathcal{D}$

$$(i)\ \|TM_w \mathbf{1}_I\|_{L^2(v)} \le A\|\mathbf{1}_I\|_{L^2(w)} \quad \text{and} \quad (ii)\ \|T^*M_v\mathbf{1}_I\|_{L^2(w)} \le A\|\mathbf{1}_I\|_{L^2(v)}$$

then:

$$\|T\|_{L^2(w^{-1})\to L^2(v)} \lesssim 2^{2r}A.$$

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Ex. The Hilbert Transform on $L^2(\mathbb{R}, \mathbb{C}^2)$ is defined by $H \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} Hf_1 \\ Hf_2 \end{bmatrix}$.

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where $\mathfrak{S} \subseteq \mathcal{D}$ and for each $I \in \mathfrak{S}$, $\sum_{J \in ch_{\mathfrak{S}}(I)} |J| \leq \frac{1}{2}|I|$, where the sum is over the maximal elements of $\mathfrak{S}$ strictly contained in I.

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Fact: All scalar CZO's can be controlled by sparse operators.

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(*i*) $\|T M_W \mathbf{1}_I e\|_{L^2(V)} \leq A \|\mathbf{1}_I e\|_{L^2(W)}$

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Remark: This theorem requires $W, V \in A_2$ and depends on $[W]_{A_2}$ and $[V]_{A_2}$.

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$$\|\{\alpha_I\}\|^2_{\ell^2(\mathcal{D},\{A_I\})} \equiv \sum_{I \in \mathcal{D}} \langle A_I \alpha_I, \alpha_I \rangle_{\mathbb{C}^d} < \infty.$$

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Matrix CET (Treil-Volberg 1997, Isralowitz-Kwon-Pott 2014)

Define the operator: $\mathcal{J} : L^2(W) \to \ell^2(\mathcal{D}, \{A_I\})$ by: $\mathcal{J} f = \{\langle W f \rangle_I\}_{I \in \mathcal{D}}$.

Key Tool: Matrix Carleson Embedding Theorem

Defn. Let $W \in A_2$ and $\{A_I\}_{I \in \mathcal{D}}$ a sequence of positive semi-definite $d \times d$ matrices. Let $\ell^2(\mathcal{D}, \{A_I\})$ be the set of $\{\alpha_I\}_{I \in \mathcal{D}}$ with $\alpha_I \in \mathbb{C}^d$ and

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Then if the following testing conditions hold:

$$\|\mathcal{J} \mathbf{1}_J e\|_{\ell^2(\mathcal{D},\{A_I\})} \leq C_T \|\mathbf{1}_J e\|_{L^2(W)} \quad \forall \; e \in \mathbb{C}^d, \; J \in \mathcal{D},$$

then the operator satisfies

$$\|\mathcal{J}\|_{L^2(W) \to \ell^2(\mathcal{D},\{A_I\})} \leq C_T C(d) [W]_{A_2}.$$

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Fact: In the scalar setting, there is no dependence on $[W]_{A_2}$.

Remaining Open Questions

- Is there a Matrix Carleson Embedding Theorem that does not require $W \in A_2$ and/or does not depend on $[W]_{A_2}$?

- Can one establish the $T(1)$ testing conditions for dyadic shifts in the matrix setting?

- Can one estimate matrix CZO's using sparse operators, as in the scalar setting?

The End!