Thin interpolating sequences

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Interpolating Sequences

H^∞ the algebra of bounded analytic functions.

Definition

A sequence (z_n) in $\mathbb{D}$ is interpolating for H^∞ if all $(w_n) \in \ell^\infty$ there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$ for all n.

Carleson's condition. A sequence (z_n) is interpolating for H^∞ iff there exists a $\delta > 0$ such that

$$\inf_j \underbrace{\prod_{k \neq j} \left| \frac{z_k - z_j}{1 - \overline{z_j} z_k} \right|}_{\rho(z_k, z_j)} \geq \delta.$$

Carleson, 1958, Amer. Math. J.

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$$B_j(z) = \prod_{k \neq j} \frac{\overline{z_k}}{|z_k|} \left(\frac{z_k - z}{1 - \overline{z_k} z} \right); \ \inf |B_j(z_j)| \geq \delta > 0, \text{ and } \delta_j = |B_j(z_j)|.$$

Shapiro-Shields, 1961

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(z_j) satisfies Carleson's condition if and only if given $(a_j) \in \ell^2$, there exists $f \in H^2$ such that $f(z_j)(1 - |z_j|^2)^{1/2} = a_j$ for all j.

Earl's Theorem

Questions: What is the norm of the best function that does the interpolation?

Can we write down a function that does the interpolation?

Earl's Theorem, J. London Math. 1970

Theorem (Earl's Theorem)

Let (z_j) be interpolating, separation const. δ. If $a = (a_n) \in \ell_\infty$ and

$$M > \underbrace{\frac{2 - \delta^2 + 2(1 - \delta^2)^{1/2}}{\delta^2}}_{M(\delta)} \|a\|_\infty$$

there exists a Blaschke prod. B such that

$$a_n = Me^{i\alpha} B(z_n), n = 1, 2, \ldots.$$

The zeros (w_j) of B can be chosen so that $\rho(w_j, z_j) \leq \frac{\delta}{1+(1-\delta^2)^{1/2}}$.

Earl's theorem rephrased: $M(\delta) \leq \left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)^2$.

Can we write down the functions that do the interpolation?

Theorem (P. Beurling)

Let (z_j) be interpolating and let

$$M = \sup_{\|a\| \leq 1} \inf\{\|f\|_\infty : f \in H^\infty, f(z_j) = a_j, j = 1, 2, 3, \ldots\}.$$

Then there are functions $f_j \in H^\infty$ such that

$$f_j(z_j) = 1, f_j(z_k) = 0, k \neq j \text{ and } \sum_j |f_j(z)| \leq M.$$

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$\sum a_j f_j(z)$ does the interpolation.

P. Beurling functions for H^∞, Peter Jones 1983

f_j with $f_j(z_j) = 1, f_j(z_k) = 0$ and $\sum_{j=1}^{\infty} |f_j(z)| < M$. What are f_j?

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$$f_j(z) := \frac{B_j(z)}{B_j(z_j)} \left(\frac{1 - |z_j|^2}{1 - \overline{z_j} z} \right)^2 e^{\left(-\frac{1}{2C(\delta)} \sum_{|z_m| \geq |z_j|} \left(\frac{1+\overline{z_m} z}{1-\overline{z_m} z} - \frac{1+\overline{z_m} z_j}{1-\overline{z_m} z_j} \right) (1-|z_m|^2) \right)}$$

Then for any $a \in \ell^\infty$, $f(z) = \sum_{j=1}^{\infty} a_j f_j(z) \in H^\infty(\mathbb{D})$

$$f(z_j) = a_j; |f(z)| \leq \|a\|_{\ell^\infty} \left(\sum_{j=1}^{\infty} |f_j(z)| \right) \leq C(\delta) \|a\|_{\ell^\infty}.$$

The constant of interpolation, Nicolau, Ortega-Cerdà, Seip, 2004;

Carleson potentials and the reproducing kernel thesis for embedding theorems, Petermichl, Treil, Wick, 2007.

Kenneth Hoffman, 1967

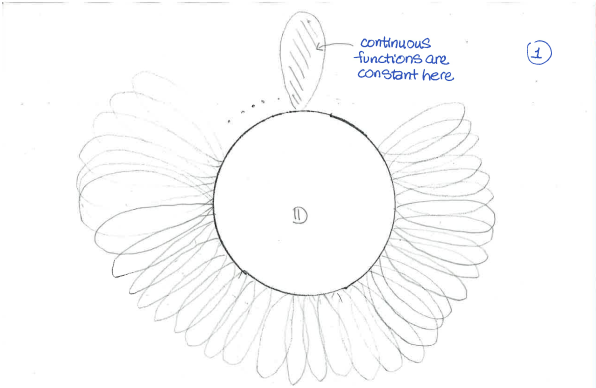
The ingredients: $H^\infty(\mathbb{D})$ and its maximal ideal space M (the set of nonzero multiplicative linear functionals on H^∞ with the weak-$\star$ topology).

Example. For $z \in \mathbb{D}$ and $f \in H^\infty$, $\varphi_z(f) = f(z)$.

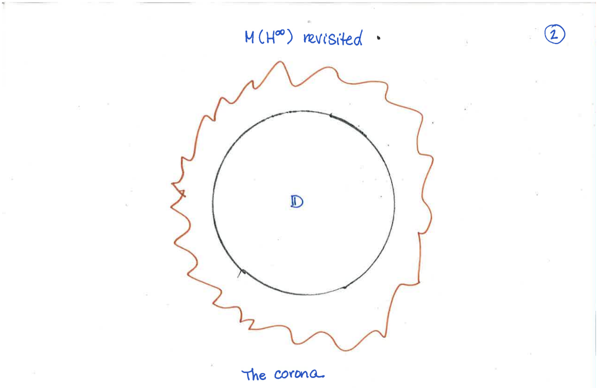
A compact Hausdorff space M such that

1. Carleson's corona theorem: M contains $\mathbb{D}$ as a dense subset;
2. Each $f \in H^\infty$ can be extended to a continuous complex function $\hat{f}$ on M (where $\hat{f}(\varphi) = \varphi(f)$).
3. Points of M are separated by functions in H^∞.

The maximal ideal space



Another view of the maximal ideal space



Gleason parts

Subsets of M that can be endowed with analytic structure.

For $m_1, m_2 \in M$ the pseudo-hyperbolic distance is

$$\rho(m_1, m_2) = \sup\{|\hat{f}(m_2)| : f \in H^\infty, \|f\|_\infty \leq 1, \hat{f}(m_1) = 0\}.$$

Gleason part of m : $P(m) = \{m_1 \in M : \rho(m, m_1) < 1\}$.

Points in $\mathbb{D} \Rightarrow \rho(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|$.

$\mathbb{D}$ is one Gleason part.

The Gleason Parts: What can we say about them?

For $\alpha \in \mathbb{D}$ let

$$L_\alpha(z) = \frac{z + \alpha}{1 + \overline{\alpha} z}.$$

$\alpha \to m$ implies $L_\alpha \to L_m$, where L_m is a map from $\mathbb{D} \to P(m)$.

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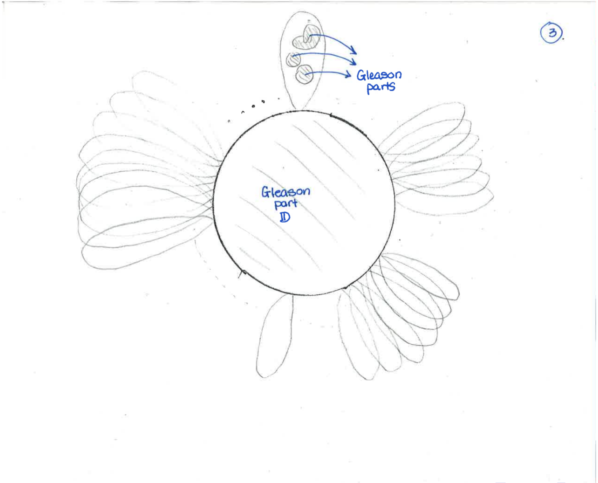
Hoffman: L_m is not constant if and only if m lies in the closure of an interpolating sequence (α_n):

$$0 < \inf_n \prod_{k \neq n} \left| \frac{\alpha_k - \alpha_n}{1 - \overline{\alpha_k} \alpha_n} \right|.$$

Picturing "analytic disks"



A special class of Blaschke products and Hoffman

$$A(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}; \;\; L_{\alpha_n}(z) = \frac{z + \alpha_n}{1 + \overline{\alpha_n} z}, \;\; L'_{\alpha_n}(z) = \frac{1 - |\alpha_n|^2}{(1 + \overline{\alpha_n} z)^2}.$$

$$|(A \circ L_{\alpha_n})'(0)| = |A'(L_{\alpha_n}(0))| L'_{\alpha_n}(0) =$$

$$(1 - |\alpha_n|^2)|A'(\alpha_n)|$$

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$(A \circ L_n)(0) = 0$ and $|(A \circ L_n)'(0)| \to 1$

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$$(A \circ L_n)(0) = 0 \text{ and } |(A \circ L_n)'(0)| \to 1 \text{ and } (\hat{A} \circ L_m)(z) = e^{i\theta} z.$$

Consequences

1) Thin parts are homeomorphic to the disk.

2) If $\delta_n(B) \to 1$ then B has at most one zero in a Gleason part. The zeros are "thin." And for any Gleason part either $B \circ L_m(z) = \lambda z$ or $|B \circ L_m| = 1$.

3) So, if we factor such B, one of the factors will have modulus 1.

and therefore $B = B_1 B_2$ one of the two must be close to one in modulus.

So the zeros of thin Blaschke products are very far apart.

Thin sequences are special

Definition

$\{\alpha_n\}$ is thin if $\lim_{n\to\infty} \prod_{k\neq n} \left|\frac{\alpha_k-\alpha_n}{1-\overline{\alpha_k}\alpha_n}\right| = 1.$

Thin sequences are indestructible: Let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Then $\varphi_a \circ B$ is thin, if the zeros of B form a thin sequence.

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Theorem (Hedenmalm)

Let E be the closed subalgebra generated by H^∞ and conjugates of all thin interpolating BPs. If u is inner and invertible in E, then u is a finite product of thin Blaschke products.

Thin sequences, Wolff, Duke 1982; Sundberg, Wolff, 1983

Sarason's algebra:
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$QC = H^\infty + C \cap \overline{H^\infty + C}$, $QA = QC \cap H^\infty$.

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Theorem (Wolff, Wolff-Sundberg)

The following are equivalent for an interpolating sequence $\{z_n\}$.

1. *For any $\{\lambda_n\} \in \ell^\infty$ there is $f \in QA$ with $f(z_n) = \lambda_n$;*
2. *For any $\{\lambda_n\} \in \ell^\infty$, $\varepsilon > 0$, then there is an $f \in H^\infty$ with $\|f\| < \limsup |\lambda_n| + \varepsilon$ and $f(z_n) = \lambda_n$ all but finitely many n.*
3. $\lim_{n\to\infty} \prod_{m\neq n} \left| \frac{z_n - z_m}{1 - \overline{z_m} z_n} \right| = 1$.

A different perspective on thin sequences

Theorem (G, Pott, Wick; consequence of Volberg)

Let $\{z_n\}$ be thin interpolating. For every $\varepsilon > 0$ there exists N such that for $n \geq N$ there exist $f_n \in H^\infty$ such that for $k \geq N$

$$f_n(z_n) = 1, f_n(z_k) = 0, n \neq k,$$

and for all $z \in \mathbb{D}$ we have

$$\sum_{n \geq N} |f_n(z)| < (1 + \varepsilon).$$

So, for all $a \in \ell^\infty$ with $\|a\|_{\ell^\infty} \leq 1$

$$g_a(z) := \sum_{n \geq N} a_n f_n(z) \in H^\infty,$$

$$\|g_a\|_\infty \leq (1 + \varepsilon)\|a\|_{N,\ell^\infty}, j \geq N, \text{ and } g_a(z_j) = a_j.$$

More ways to think about thin: $\lim_n \delta_n = 1$

1) (GM) A sequence (z_n) is thin iff whenever $a \in$ ball (ℓ^∞) there exists $f \in H^\infty$ with $\|f\|_\infty \leq 1$ such that $|f(z_j) - a_j| \to 0$. (AIS)

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2.(GPW) **Definition (interpolation, good bound)**: $\{z_n\}$ is an eventually $(1+\varepsilon)$ interpolating for H^p (EIS_p) if: For every $\varepsilon > 0$ there exists N such that for each $\{a_n\} \in \ell^p$ there exists $f_{N,a} \in H^p$ with

$$f_{N,a}(z_n)(1-|z_n|^2)^{1/p} = a_n \text{ for } n \geq N \text{ and } \|f_{N,a}\|_p \leq (1+\varepsilon)\|a_n\|_{N,\ell^p}.$$

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3. **Definition (asymptotic, best bound)**: $\{z_j\}$ is an AIS_p-sequence if for all $\varepsilon > 0$ there exists N such that for all $\{a_j\} \in \ell^p$ there exists $G_{N,a} \in H^p$ such that $\|G_{N,a}\|_p \le \|a\|_{N,\ell^p}$ and

$$|G_{N,a}(z_j)(1-|z_j|^2)^{1/p} - a_j|_{N,\ell^p} < \varepsilon \|a_j\|_{N,\ell^p}.$$

Theorem: These are equivalent to thin.

Dyakonov, Nicolau, 2007 TAMS

Theorem

Let (z_n) be interpolating. The following are equivalent:

1. (z_j) *is thin; i.e.* $\lim_{k\to\infty} \prod_{j\neq k} \left| \frac{z_j - z_k}{1-\overline{z_j} z_k} \right| = 1.$
2. *There is a sequence* (m_j), $0 < m_j < 1$ *and* $m_j \to 1$ *such that every interpolation problems* $F(z_j) = w_j$ *with* $|w_j| \leq m_j$ *has a solution* $f \in H^\infty$ *with* $\|F\| \leq 1$.
3. *There exists* (ε_j) *such that every interpolation problem with* $1 \geq |a_j| \geq \varepsilon_j$ *for all* j *has a nonvanishing solution* $g \in H^\infty$.

Operator theory and thin sequences

Brown and Halmos

For $\varphi \in L^\infty$ define the Toeplitz operator on H^2 by

$$T_\varphi f = P\varphi f, P \text{ the orthogonal projection from } L^2 \text{ to } H^2.$$

The Hankel operator is

$$H_\varphi f = (I - P)\varphi f,\ f \in H^2.$$

Theorem (Brown, Halmos, 1964)

Let f, g be in L^∞. Then $T_f T_g = T_{fg}$ if and only if $\overline{f} \in H^\infty$ or $g \in H^\infty$.

Question. For which symbols f, g is $T_f T_g$ a compact perturbation of a Toeplitz operator, (T_{fg})?

For which symbols, f, g is $T_{fg} - T_f T_g = H^\star_{\bar{f}} H_g$ compact?

Operator theory again

Theorem (Axler, Chang, Sarason 1978 (IEOT))

If

$$H^\infty[f] \cap H^\infty[g] \subset H^\infty + C,$$

then $H_f^\star H_g$ is compact.

Necessity was proved for a large class of functions. Proof involved looking at the support set of elements of $M(H^\infty)$.

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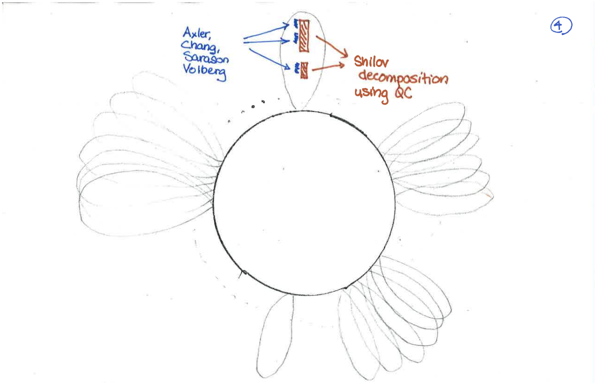
Volberg, Part 1, 1982 (JOT)

Theorem

Suppose that $H_u^\star H_v$ is compact. Then

$$H^\infty[u] \cap H^\infty[v] \subset H^\infty + C.$$

One more look at the maximal ideal space



Stein and Halmos

Key step: (E. Stein) Let $d\mu$ be a positive measure and v and w positive measurable functions on X. Suppose S is a linear operator on $L^2(vd\mu)$ and $L^2(wd\mu)$ with norms $\|S\|_v$ and $\|S\|_w$. If $\|S\|_{\sqrt{vw}}$ is the norm of S on $L^2(\sqrt{vw}d\mu)$, then

$$\|S\|_{\sqrt{vw}} \leq \sqrt{\|S\|_v \|S\|_w}.$$

Halmos, 1982 (JOT) provided new proof of Stein's result:

"The purpose of this note is to make that [Volberg's] solution more accessible by offering a simple interpretation..."

Definition: H a Hilbert space, (x_n) sequence in H.

Riesz if there exist c and C positive such that

$$c \sum_{n \geq 1} |a_n|^2 \leq \| \sum_{n \geq 1} a_n x_n \|^2 \leq C \sum_{n \geq 1} |a_n|^2;$$

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and AOS if there exists N_0 such that for $N \geq N_0$ there exist $c_N \to 1, C_N \to 1$ such that

$$c_N \sum_{n\geq N} |a_n|^2 \leq \| \sum_{n\geq N} a_n x_n \|^2 \leq C_N \sum_{n\geq N} |a_n|^2.$$

Volberg, Part 2: Riesz sequences and AOS

Definition: H a Hilbert space, (x_n) sequence in H.

Riesz if there exist c and C positive such that

$$c\sum_{n\geq 1}|a_n|^2 \leq \|\sum_{n\geq 1} a_n x_n\|^2 \leq C\sum_{n\geq 1}|a_n|^2;$$

and AOS if there exists N_0 such that for $N \geq N_0$ there exist $c_N \to 1, C_N \to 1$ such that

$$c_N\sum_{n\geq N}|a_n|^2 \leq \|\sum_{n\geq N} a_n x_n\|^2 \leq C_N\sum_{n\geq N}|a_n|^2.$$

Gram matrix of (x_n) is $G = (\langle x_n, x_m\rangle)_{n,m}$

Riesz sequences are the ones for which G is invertible.

What are the AOS (almost orthogonal sequences)?

Let $g_\lambda(z) = \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z}$. The $\{g_{\lambda_n}\}$ form a Riesz basis for their span if and only if

$$A(z) = \prod_{n=1}^{\infty} \frac{\bar{\lambda_n}}{|\lambda_n|} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z}$$

satisfies

$$\inf_n \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda_k} \lambda_n} \right| > 0.$$

Question. When is (g_{λ_n}) near an orthogonal basis?: i.e., $g_{\lambda_n} = (U + K)e_n$ where (e_n) is the standard orthonormal basis of ℓ^2, U is unitary and K is compact.

Volberg's Second Theorem

Theorem

(g_{λ_n}) near an orthogonal basis ($g_n = (U+K)(e_n)$) iff (λ_n) is thin (that is, $\lim_{n\to\infty} \delta_n = 1$ where $\delta_n = \prod_{k\neq n} \left|\frac{\lambda_k - \lambda_n}{1-\overline{\lambda_k}\lambda_n}\right|$).

The proof: There are two absolute constants C_1 and C_2 so that if B_1, B_2 are Blaschke with $\inf \max(|B_1|, |B_2|) \geq \delta$ then $\|H^{\star}_{\overline{B_1}} H_{B_2}\| \leq C_1(1-\delta)^{C_2}$.

Let B be Blaschke with zeros (λ_n) and δ as above. Then for every factorization $B = B_1 B_2$ we have

$$\inf_{\mathbb{D}} \max(|B_1|, |B_2|) \geq \eta^2,$$

where $\eta = \frac{1-\sqrt{1-\delta^2}}{\delta}$.

AOS if there exists N_0 such that for $N \geq N_0$ there exists $c_N \to 1$, $C_N \to 1$ such that

$$c_N \sum_{n \geq N} |a_n|^2 \leq \| \sum_{n \geq N} a_n x_n \|^2 \leq C_N \sum_{n \geq N} |a_n|^2.$$

Theorem

Let (x_n) be a sequence in a (separable) H. Then (x_n) is AOS and x_n is not in the span of $\{x_m\}_{m \neq n}$ iff $G = I + K$ with K compact defines a bounded invertible operator.

Introducing the Gram matrix

Recall that k_j denotes the Szegő kernel, $g_j = k_j/\|k_j\|$, and G the Gram matrix with entries $G_{ij} = \langle g_j, g_i \rangle$.

For $\{\alpha_j\}$ interpolating, we let D be the diagonal matrix with entries $1/B_j(\alpha_j)$.

Fact:

$$G^{-1} \;=\; D^* G^t D.$$

We need

$$\|G - I\| \text{ and } \|G\| \text{ or } \|G^{-1}\|.$$

Shapiro, Shields, and Earl

Proposition (Shapiro, Shields)

Let $\{\alpha_j\}$ be an interpolating sequence in $\mathbb{D}$.
(i) If the interpolation constant is M, then both $\|G\|$ and $\|G^{-1}\|$ are bounded by M^2.
(ii) If $\|G\| = C_1$ and $\|G^{-1}\| = C_2$, then the interpolation constant is bounded by $\sqrt{C_1 C_2}$.

Theorem (Earl's Theorem)

The interpolation constant $M(\delta)$ satisfies

$$M(\delta) \leq \left(\frac{1 + \sqrt{1 - \delta^2}}{\delta} \right)^2 .$$

Another proof of Volberg's result

Theorem (Volberg's, via G., McCarthy, Pott, Wick)

The sequence $\{\alpha_n\}$ is a thin sequence if and only if the Gram matrix G is the identity plus a compact operator.

Fact: Let $\{\alpha_j\}$ be interpolating and G the Gram matrix.

Let $C = \|G^{-1}\|$. Then $\|G - I\| \leq C - 1$.

Show $G = I + K \Rightarrow$ thin

$$G^{-1} - I = D^* G^t D - I = D^*(G^t - I)D + [D^* D - I].$$

$G - I$ compact $\Rightarrow$ $G^t - I$ and $G^{-1} - I = G^{-1}(I - G)$ compact.

So $D^* D - I$ is compact.

Diagonal entries are $(1/\overline{B_j(\alpha_j)}) \cdot (1/B_j(\alpha_j)) - 1 =$

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So $D^* D - I$ is compact.

Diagonal entries are $(1/\overline{B_j(\alpha_j)}) \cdot (1/B_j(\alpha_j)) - 1 = 1/\delta_j^2 - 1$.

So $\lim_{j\to\infty} \delta_j^2 = 1$. Consequently, the sequence is thin.

Sketch of “thin $\Rightarrow G = I + K$”

$\{\alpha_n\}$ is thin. Assume $\{\alpha_n\}$ interpolating.

Let G_N be the Gram matrix of $\{g_j\}$ for $j \geq N$

By Earl's theorem and fact: $C = \|G^{-1}\|$ implies $\|G - I\| \leq C - 1$.
So

$$\|G_N^{-1}\| \leq (M(\delta(N))^2$$

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$$\|G_N^{-1}\| \leq (M(\delta(N))^2 \leq \frac{\left(1 + \sqrt{1 - \delta(N)}\right)^4}{\delta(N)^4}$$

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$$\Rightarrow \|G_N - I_N\| \leq \left(\frac{\left(1 + \sqrt{1 - \delta_N^2}\right)^4}{\delta_N^4} - 1 \right)$$

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By Earl's theorem and fact: $C = \|G^{-1}\|$ implies $\|G - I\| \leq C - 1$. So

$$\|G_N^{-1}\| \leq (M(\delta(N))^2 \leq \frac{\left(1 + \sqrt{1 - \delta(N)}\right)^4}{\delta(N)^4} \leq \frac{\left(1 + \sqrt{1 - \delta_N^2}\right)^4}{\delta_N^4}.$$

$$\Rightarrow \|G_N - I_N\| \leq \left(\frac{\left(1 + \sqrt{1 - \delta_N^2}\right)^4}{\delta_N^4} - 1 \right) \leq C\sqrt{1 - \delta_N},$$

where C independent of N. But $\sqrt{1 - \delta_N} \to 0$ as $N \to \infty$, so $G - I$ is compact.

S_∞ = compact operators

Up to now we have looked at when $\{g_\lambda\}$ is a $U + S_\infty$ basis.

The Schatten-p classes: Compact operators for which the corresponding singular sequence is in ℓ^p: $\|T\|_p = (\sum_n |\lambda_n|^p)^{1/p}$.

Letting S_2 denote the Hilbert-Schmidt operators, Volberg also showed that

$$\{g_{\lambda_n}\} \text{ is a } U + S_2 \text{ basis if and only if } \prod_{n\geq 1} \delta_n \text{ converges.}$$

Volberg covered $p = \infty$ and $p = 2$. What about the other values?

Schatten-p class

Theorem

Let T be an operator on a separable Hilbert space, $\mathcal{H}$.

$0 < p \leq 2 \implies \|T\|_{\mathcal{S}_p}^p =$
$\inf\left\{\sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H}\right\}$

$2 \leq p < \infty \implies \|T\|_{\mathcal{S}_p}^p =$
$\sup\left\{\sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H}\right\}.$

Theorem

Let $\{e_n\}$ denote the standard orthonormal basis for ℓ^2 and $\{\alpha_j\}$ be an interpolating sequence in $\mathbb{D}$ with corresponding δ_j. Then

$$\|(G - I)e_n\| \asymp \sqrt{1 - \delta_n^2}.$$

Putting this together

Theorem

The following estimates hold:

- *If* $2 \le p < \infty$ *then*

$$\sum_n (1-\delta_n)^{\frac{p}{2}} \lesssim \|G - I\|_{\mathcal{S}_p}^p;$$

- *If* $0 < p \le 2$ *then*

$$\|G - I\|_{\mathcal{S}_p}^p \lesssim \sum_n (1-\delta_n)^{\frac{p}{2}};$$

- *If* $p = 2$ *then*

$$\sum_n (1-\delta_n) \asymp \|G - I\|_{\mathcal{S}_2}^2.$$

G., McCarthy, Pott, Wick

Theorem

For $2 \leq p < \infty$, $G - I \in S_p$ if and only if $\sum_n (1 - \delta_n^2)^{p/2} < \infty$.

Lopatto and Rochberg have looked at this for truncated Toeplitz operators on model spaces.