Duality and peak interpolation for multipliers of the Drury-Arveson space

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BIRS Multivariate Operator Theory
April 9, 2015

Joint work with Ken Davidson.

Henkin measures

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Let $H^\infty(\mathbb{B}_d)$ be the algebra of bounded holomorphic functions on $\mathbb{B}_d$.
Both of these algebras are equipped with the norm

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The algebra $H^\infty(\mathbb{B}_d)$ has a weak-$*$ topology which can be described as follows: a sequence $\{f_n\}_n \subset H^\infty(\mathbb{B}_d)$ converges to 0 in the weak-$*$ topology if and only if it is bounded and $f_n(z) \to 0$ for every $z \in \mathbb{B}_d$.

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Definition

We say that a measure μ on the sphere $\mathbb{S}_d$ is $A(\mathbb{B}_d)$–*Henkin* if

$$\int_{\mathbb{S}_d} f_n d\mu \to 0$$

whenever $\{f_n\}_n \subset A(\mathbb{B}_d)$ is a bounded sequence converging to 0 pointwise on $\mathbb{B}_d$.

Some examples

- Let σ denote normalized Lebesgue measure on $\mathbb{S}_d$. By the Cauchy formula,

$$f(0) = \int_{\mathbb{S}_d} f d\sigma$$

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- Recall that a measure μ on $\mathbb{S}_d$ is called a *representing measure for* $\lambda \in \mathbb{B}_d$ if

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- (Henkin) If μ is absolutely continuous with respect to an $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin measure, then μ is $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin itself.

Connection with single operator theory

Let $T \in B(\mathcal{H})$ be a contraction. When can we extend the polynomial functional calculus

$$p \mapsto p(T)$$

to a contractive homomorphism

$$\Phi_T : H^\infty(\mathbb{D}) \to B(\mathcal{H})$$

which is weak-$*$ continuous?

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$$T = T_{cnu} \oplus U$$

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QUESTION Can we characterize $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin measures?

The characterization

Denote by $M_0(\mathbb{S}_d)$ the set of positive representing measures $\mu \in M(\mathbb{S}_d)$ for the origin, that is

$$\int_{\mathbb{S}_d} f d\mu = f(0)$$

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We have $M_0(\mathbb{T}) = \{\sigma\}$. In higher dimensions, we have $\sigma \in M_0(\mathbb{S}_d)$ but there are other measures as well.

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Theorem (Henkin, Cole-Range)

A measure $\mu \in M(\mathbb{S}_d)$ is $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin if and only it is absolutely continuous with respect to some measure in $M_0(\mathbb{S}_d)$.

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we obtain that

$$(A(\mathbb{B}_d))^* = (H^\infty(\mathbb{B}_d))_* \oplus_1 TS(\mathbb{S}_d)$$

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where $TS(\mathbb{S}_d)$ is the space of measures on $\mathbb{S}_d$ which are singular with respect to every measure in $M_0(\mathbb{S}_d)$.
Such measures are said to be $\mathrm{A}(\mathbb{B}_\mathrm{d})$-*totally singular*.

Application to multivariable operator theory

Let $T_1, \ldots, T_d \in B(\mathcal{H})$ be commuting operators such that $T = (T_1, \ldots, T_d)$ is a contraction, which is equivalent to

$$\sum_{k=1}^{d} T_k T_k^* \leq I.$$

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Here $\mathcal{M}_d$ is the multiplier algebra $\mathcal{M}_d$ of the Drury-Arveson space H^2_d.
(The reproducing kernel Hilbert space on $\mathbb{B}_d$ with kernel $k(z,w) = (1 - \langle z, w \rangle)^{-1}$ and $\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \alpha! / |\alpha|!$)

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The problem now becomes to characterize the following functionals.

Definition

We say that a functional $\Psi \in \mathcal{A}_d^*$ is $\mathcal{A}_d$*–Henkin* if $\Psi(f_n) \to 0$ whenever $\{f_n\}_n \subset \mathcal{A}_d$ converges to 0 in the weak-$*$ topology of $\mathcal{M}_d$.

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Since $\|f\|_\infty \leq \|f\|_{\mathcal{M}_d}$, the condition of μ being $\mathcal{A}_d$–Henkin is a priori weaker than μ being an $A(\mathbb{B}_d)$–Henkin.
Therefore, the Henkin-Cole-Range characterization does not apply to $\mathcal{A}_d$–Henkin measures.

Some new results

Theorem (C.-Davidson 2015)

There is a commutative von Neumann algebra $\mathfrak{W}$ *such that*

$$\mathcal{A}_d^{**} \cong \mathcal{M}_d \oplus \mathfrak{W}$$

and in particular

$$\mathcal{A}_d^* \cong (\mathcal{M}_d)_* \oplus_1 \mathfrak{W}_*.$$

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Let $\Psi \in \mathfrak{W}_*$. *Then there exists an* $\mathrm{A}(\mathbb{B}_\mathrm{d})$*–totally singular measure* $\mu \in M(\mathbb{S}_d)$ *with* $\|\mu\| = \|\Psi\|$ *such that*

$$\Psi(f) = \int_{\mathbb{S}_d} f d\mu, \quad f \in \mathcal{A}_d\,.$$

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Accordingly, a measure $\mu \in M(\mathbb{S}_d)$ for which $\Psi \in \mathfrak{W}_*$ will be called $\mathcal{A}_d$*–totally singular*.

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- A measure $\mu \in M(\mathbb{S}_d)$ is $\mathrm{A}(\mathbb{B}_\mathrm{d})$–totally singular if and only if it is $\mathcal{A}_d$–totally singular.
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A family of measures $\mathcal{B}$ is called a *band* if $\mu \in \mathcal{B}$ whenever $\mu_0 \in \mathcal{B}$ and μ is absolutely continuous with respect to μ_0.

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$\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin measures form a band (Henkin) and so do the $\mathrm{A}(\mathbb{B}_\mathrm{d})$–totally singular measures (trivial).

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The following supports the previous conjecture.

Theorem (C.-Davidson 2015)

Let $\mu_1, \mu_2 \in M(\mathbb{S}_d)$ and such that μ_1 is absolutely continuous with respect to μ_2.

- *If μ_2 is $\mathcal{A}_d$–Henkin then so is μ_1.*
- *If μ_2 is $\mathcal{A}_d$–totally singular, then so is μ_1.*

Peak interpolation for uniform algebras

Let X be a compact Hausdorff space and let $A \subset C(X)$ be a unital closed subalgebra (a "uniform algebra"). A closed subset $K \subset X$ is called a *peak interpolation set for* A if for every $h \in C(K)$ there exists $f \in A$ such that

- $f = h$ on K
- $|f(x)| < \|h\|_K$ for every $x \in X \setminus K$.

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Theorem (Carleson-Rudin, Bishop)

Let $K \subset \mathbb{S}_d$ be a closed subset such that $|\mu|(K) = 0$ for every $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin measure μ. Then, K is a peak interpolation set for $A(\mathbb{B}_d)$.

Peak interpolation for $\mathcal{A}_d$?

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Theorem (C.-Davidson 2015)

Let $K \subset \mathbb{S}_d$ be a closed subset with the property that $|\mu|(K) = 0$ for every $\mathcal{A}_d$–Henkin measure μ. Then, for every $h \in C(K)$ and every $\varepsilon > 0$ there exists $f \in \mathcal{A}_d$ such that

- $f = h$ on K
- $|f(\zeta)| < \|h\|_K$ for every $\zeta \in \mathbb{S}_d \setminus K$
- $\|f\|_{\mathcal{A}_d} \leq (1+\varepsilon)\|h\|_K$.

Peak interpolation for $\mathcal{A}_d$?

In general $\|f\|_\infty < \|f\|_{\mathcal{M}_d}$ so $\mathcal{A}_d$ cannot be viewed as a closed subalgebra of $C(\mathbb{S}_d)$. By considering the canonical isometric inclusion $j : \mathcal{A}_d \to \mathcal{A}_d^{**}$, we can view $j(\mathcal{A}_d)$ as closed subspace of $C(X)$, where X is the unit ball of $\mathcal{A}_d^*$ equipped with the weak-$*$ topology. We can then attempt to apply a general result of Bishop to that situation to achieve peak interpolation on a set $K \subset \mathbb{S}_d$.

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The condition is certainly satisfied for countable sets. Note that achieving interpolation (let alone peak interpolation) in $\mathcal{A}_d$ on such a set isn't trivial!

Ingredients of the proof: Choquet integral representation and concave functions

Recall that X is the unit ball of $\mathcal{A}_d^*$ equipped with the weak-$*$ topology.

Theorem (Bishop-de Leeuw 1959, Hustad 1971)

Let $B \subset C(X)$ be a closed subspace containing the constant functions and $\Phi \in B^*$. Then, there exists a measure μ concentrated on the extreme points of X with $\|\mu\| = \|\Phi\|$ such that

$$\Phi(b) = \int_X b d\mu, \quad b \in B.$$

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Theorem (C.-Davidson 2015)

Let $\Phi \in B_t^*$ and assume that t is concave. Then, there exists a measure μ concentrated on the extreme points of X with $\|\mu\| = \|\Phi\|$ such that

$$\Phi(b/t) = \int_X \frac{b}{t} d\mu, \quad b \in B.$$

Thank you!