Lifting questions

Frans Oort

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Consider

$$\begin{array}{ccccc} X_0 & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_\eta \\ \downarrow & & \downarrow & & \downarrow \\ \{s_0\} = \mathrm{Spec}(\kappa) & \hookrightarrow & S & \hookleftarrow & \mathrm{Spec}(K) = \{\eta\} \end{array}$$

Here S is an integral scheme, with a closed point $s_0 \in S$ and generic point η. Further $\mathcal{X}/S$ is a morphism, perhaps with additional structures on $\mathcal{X}$, possibly with conditions on the morphism $\mathcal{X} \to S$ (such as flat and several other conditions, depending on the situation). In this diagram both squares are cartesian, i.e. $X_0 = \mathrm{Spec}(\kappa) \times_S \mathcal{X}$ and $X_\eta = \mathrm{Spec}(K) \times_S \mathcal{X}$.

We say that the *special fiber* X_0 is a specialization of the *generic fiber* X_η.

If X_η is given, finding such a diagram with certain properties is called "reduction theory". Before scheme-theory was invented this was often a practice difficult to describe. E.g. compare the seminal original paper by Néron on *minimal models*, see [17], and later understanding of this topic, such as [1]. – However, the theory of "good and bad reductions" will not be the focus of this talk. We will consider the opposite:

Suppose X_0 given and ask which diagrams as above are possible. Let us agree on *terminology* that

- If $p = \mathrm{char}(\kappa) > 0 = \mathrm{char}(K)$ we say that X_η is a *lifting* (to characteristic zero) of X_0.
- If $\mathrm{char}(\kappa) = \mathrm{char}(K)$ we say that X_η is a *deformation* of X_0

(of course conceptually the same, but let us use this terminology to make life easier). And:

- If the result $\mathcal{X}/S$ is a formal scheme we say a formal deformation, or a formal lifting.
- If the result $\mathcal{X}/S$ is an algebraic scheme (with possibly extra structure) we say a deformation, or a lifting.

- Suppose in the diagram above $S = \mathrm{Spec}(R)$; it might be this is possible with *normal domain* R. However (in the second talk this week) we will encounter a situation where a lifting to a normal domain is not possible, but where is a lifting is possible (how do you find an invariant to test this kind of possibilities ?).
- Suppose you want to show that in a certain situation a lifting (to characteristic zero) is not possible, how do you prove that? what kind of invariant would be suitable to test this?
- Sometimes lifting to an unramified ring is possible, sometimes ramification is needed (we will see an example).

Amenability. In this field impressive progress has been made. However, to understand the beauty and the impact of this part of mathematics it is better not to stare at results, but to start thinking about questions and methods. Therefore, in this introductory talk I will

formulate questions

and let you think about

possible answers

(many of these will be given in this week) (with apologies to all experts present, who know this material so well).

When saying "question" or "exercise" below this is just meant to stimulate the reader to start thinking about this topic; there is no guarantee the problem is solved or not; there is no guarantee the exercise is easy (for some a solution exist, but you are a genius if you find it in short time).

In this introduction we only treat a small part of all deformation and lifting problems (and some of the choices are triggered by my personal taste).

In the rich spectrum of topics we will discuss only:

(1) Liftings and deformations of algebraic curves.

(2) Liftings of algebraic curves and automorphisms.

(3) Deformations of curves and Newton polygons.

(4) Liftings of higher dimensional varieties.

(5) Liftings of abelian varieties.

(6) Deformations of abelian varieties with given Newton polygons

(7) CM liftings of abelian varieties.

Notation. We write k for an algebraically closed field. We write K for a field, and κ for a field (usually a residue class field, most of the times a field in positive characteristic. When saying "an algebraic curve" over a field, we will intend a non-singular, absolutely irreducible, complete algebraic curve, unless otherwise stated. An elliptic curve E over a field K is a curve of genus one with a chosen rational point $0 \in E(K)$ (please do not give in to the bad habit of calling every curve of genus one an elliptic curve).

Methods

(0.1) Easy: equations. Suppose $C \subset \mathbb{P}^1_\kappa$ is a *plane curve*; the study of liftings and deformations as a plane curve is easy (by lifting the coefficients in a defining equation to some ring $R \twoheadrightarrow \kappa$). (More generally this applies to hypersurfaces.)
Exercise. Show any elliptic curve defined over any $\kappa \supset \mathbb{F}_p$ can be lifted to characteristic zero.
Question. Let E be an elliptic curve defined over any $\kappa \supset \mathbb{F}_p$. Does a lift of $(E, \mathrm{End}(E))$ to characteristic zero exist ? (Hint: find a counterexample.)
Question. Let E be an elliptic curve defined over any $\kappa \supset \mathbb{F}_p$, and some $b \in \mathrm{End}(E)$. Does a lift of (E, b) to characteristic zero exist ?

Experience shows that on the one hand only working with equations is not sufficient in many problems. On the other hand as soon as general methods do not give an answer we are looking for, sometimes we have to do computations, consider equations, coordinates etc. We will see examples, and we discuss this aspect below.

As you see, many aspects will not be discussed (lifting of K3 surfaces, deformations of singular curves, and many other interesting cases).

Here is the fist question: let C be an algebraic curve over any $\kappa \supset \mathbb{F}_p$. Does a lift of C to characteristic zero exist? How do you approach this question? (Probably you do not want to write equations in the general situation ?)

Kodaira-Spencer

(0.2) Deformation theory. In 1958-1960 Kodaira and Spencer wrote a series of influential papers on deformation theory. Such ideas where around at that time, e.g. see [14]. If you want a start on this topic: first read these papers and try to contemplate how to generalize, how to make considerations more inside the frame work of algebraic geometry.

In the Kodaira-Sencer theory we see that the general ideas are present. However in their theory they presented a mixture of structures: often the base space was a differentiable manyfold, and the family a kind of mixture of various geometric structure, with complex analytic structures on the fibers; ideas were clear, but applications e.g. to mixed characteristic situations where hard. Please remember that in this days an object like $\mathrm{Spec}(K[\epsilon]/(\epsilon^2)$ did not exist. (I want to say, we did not know it.) – Around that time, but before Grothendieck taught us how to do algebraic geometry, difficult considerations were developed (e.g. in order to handle bad/good reduction of algebraic varieties modulo p).

Grothendieck and Schlessinger

Grothendieck tried to formulate a general deformation theory, however his approach was not successful (and luckily now forgotten). However, in his Harvard PhD-thesis M. Schlessinger formulated a beautiful approach, see [25] (only possible after we had scheme theory):

(0.3) Infinitesimal deformations. Suppose give a field κ and X_0 over κ (think of an algebraic curve, a singular curve, an algebraic variety, an abelian variety, a scheme equipped with additional structure, just choose your favorite topic).
Equal characteristic. Choose the category Λ_κ of aritinian local κ-algebras with a given

κ-homomorphism $R \twoheadrightarrow \kappa$.

Unequal characteristic. Suppose $\kappa \supset \mathbb{F}_p$ is a *perfect* field, and let $W = W_\infty(\kappa)$ be the ring of infinite Witt vectors with coordinates in κ (e.g. if $\kappa = \mathbb{F}_p$, the ring $W \cong \mathbb{Z}_p$ is the ring of p-adic integers). Choose the category Λ_κ of aritinian local W-algebras with a given W-homomorphism $R \twoheadrightarrow \kappa$.

Consider the category of $\mathcal{X} \to \mathrm{Spec}(R)$ with a given identification $\mathcal{X} \otimes \kappa \cong X_0$ with all structures involved. Formulate the obvious isomorphism concept for such objects. This gives a covariant functor, contravariant on schemes like $\mathrm{Spec}(R)$:

$$\mathcal{D}_{X_0} : \Lambda_\kappa \longrightarrow \mathrm{Sets}.$$

In many situations this functor is *pro-representable*. This is the correct generalization of Kodaira-Spencer theory inside algebraic geometry. (Even in complex geometry this gives new ideas and results.)

(0.4) Contemplate about deformations of complex tori, like $T_0 = \mathbb{C}^g/(\text{a lattice})$ and Schlessinger theory for an abelian variety A_0 over $\mathbb{C}$, and compare theories under $T_0 = A_0(\mathbb{C})$. Show, in two ways, there exist a complex 2-dimensional torus (the "generic fiber" in the deformation theory just described) not algebraizable (i.e. it is not the torus of complex points of an abelian variety).

(0.5) Say X_0 is a complete non-singular variety over κ. Denote by

$$\Theta_{X_0} = \mathcal{T}_{X_0} = \mathcal{HOM}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$$

the "tangential sheaf" on X_0; e.g. see [10], II.8.

The deformation theory is described by the following observations; let $R \twoheadrightarrow R'$ be a "small surjection" in Λ_κ (small: the kernel $I \subset R'$ has the property (max. ideal of R).$I = 0$); for every $X' \in \mathcal{D}_{X_0}(R')$

the fiber of the map $\mathcal{D}_{X_0}(R) \to \mathcal{D}_{X_0}(R')$ is (either empty or) principally homogenous under $\mathrm{H}^1(X_0, \mathcal{T}_{X_0}) \otimes I$,

and

the deformation problem is unobstructed if $\mathrm{H}^2(X_0, \mathcal{T}_{X_0}) = 0$.

We see a beautiful and direct generalization of the Kodaira-Spencer theory. For results and surveys see [5], Exp. III, see [7], 182-12/13, 195-19, 236-19/20, see [9], Chapter 6, see [19], Section 2.

1 Liftings and deformations of algebraic curves

We show that the theory just exposed solves all problems on deforming and lifting (non-singular !) algebraic curves. The local moduli problem is pro-representable, it is unobstructed, because $\mathrm{H}^2(X_0, \mathcal{T}_{X_0}) = 0$, hence the local module space gives a formal curve over a formal power series ring (over K in deformation theory, over W in mixed characteristic deformation theory). Moreover (this will be discussed later in more generality) this formal curve carries an effective divisor, and a multiple of this is very ample, hence by (Chow-GAGA-Grothendieck), to be discussed later, this formal curve comes from a (family of) algebraic curve(s). This last step is the analogue, generalization of the classical theorem by Riemann that any Riemann surface is algebraizable, e.g. see [10], Appendix B.3.

We have complete insight in the problem in this case. The (relative) dimension of the local deformation space can be computed from the dimension of $\mathrm{H}^1(X_0, \mathcal{T}_{X_0})$ and this dimension is 0 (in genus 0), 1 (for elliptic curves) and equal to $3g-3$ (for genus at least two) as Riemann already computed (over $\mathbb{C}$) in 1857.

After Grothendieck and Mumford constructed fine module spaces for algebraic curves with level structure we know that these spaces are non-singular (respectively smooth over $\mathrm{Spec}(\mathbb{Z})$) of the given (relative) dimension.

(1.1) Conclusion. The general theory of deformations and liftings of algebraic curves follows from general principles, the problem is unobstructed, and most questions have clear answers.

2 Liftings of algebraic curves and automorphisms

This will be one of the main themes of this conference.

(2.1) Exercise (Hurwitz). Let C be an algebraic curve of genus $g \geq 2$ over a field of characteristic zero. Show that

$$\#\left(\mathrm{Aut}(C)\right) \leq 84(g-1).$$

(2.2) Exercise (Roquette 1970). Consider the curve C given as complete non-singular model of $Y^2 = X^p - X$ over $\overline{\mathbb{F}_5}$, here $p = 5$. Compute the genus of C and compute $\#\left(\mathrm{Aut}(C)\right)$.

(2.3) Exercise (1). Let C_0 be as in the previous exercise. Does $(C_0, \mathrm{Aut}(C_0))$ lift to characteristic zero?
(2). Let $N \subset \mathrm{Aut}(C_0)$ be the subgroup generated by

$$\beta = (X \mapsto X+1, Y \mapsto Y) \quad \text{and} \quad \gamma = (X \mapsto -X, Y \mapsto 2Y);$$

check these two indeed define automorphisms, and compute the order of $N = < \beta, \gamma >$. (Observe that $\#(N) < 84(g-1)$.) Does the pair (C_0, N) admit a lift to characteristic zero?

(2.4) Exercise (1). Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let $n \in \mathbb{Z}_{\geq 0}$. show there exists a subgroup $G_n \subset \mathbb{P}^1_k$ with $G_n \cong (\mathbb{Z}/p)^n$.
(2) Does the pair $(\mathbb{P}^1_k, G_n)$ admit a lifting to characteristic zero for $n = 2$? (Does the answer depend on the choice of p?)

(2.5) Question. For which curves C_0/κ and which $G \subset \mathrm{Aut}(C_0)$ is a lifting to characteristic zero possible?

Comment. If $\mathrm{char}(\kappa) = p > 0$ and p does not divide the order of G the deformation / lifting problem of the pair (C_0, G) is unobstructed. In characteristic zero this is related to Riemann's uniqueness theorem.

(2.6) Problem. After you have digested methods and results on this topic, try to answer: is there an example of an algebraic curve C_0 in positive characteristic (over a non-algebraically closed field) and $b \in \mathrm{Aut}(C_0)$ such that the pair (C_0, b) *cannot be lifted to a normal domain*?

3 Deformations of curves, of abelian varieties and Newton polygons

(3.1) For an abelian variety A of dimension g over a field $\kappa \supset \mathbb{F}_p$ once can define its Newton Polygon $\mathcal{N}(A)$. In case A is defined over $\mathbb{F}_p$ this is exactly the NP of the Frobenius endomorphism $F : A \to A^{(p)} \cong A$. For other fields the definition is somewhat more complicated to give. This invariant is a lower convex polygon in $\mathbb{R} \times \mathbb{R}$, starting as $(0, 0)$, ending at $(2g, g)$, and having integral breakpoints (and other properties). It is invariant under isogenies. It is one of the first invariants describing properties of the p-structure of A. As an example: for an elliptic curve E in characteristic p its NP either consists of

two edges both having slope equal to 1/2 (the NP is a straight line),
and this is the case iff E is *supersingular*, or

the two edges have slope 0 and 1, and this is the case iff E is *ordinary*.

We say a N is symmetric if the slopes $0 \leq \lambda \leq 1$ and $1 \geq 1 - \lambda \geq 0$ appear with the same multiplicity. It can be shown that for an abelian variety its NP is symmetric.

(3.2) **A conjecture by Manin,** see [15], page 76, Conjecture 2. Is it true that for every p every possible symmetric NP is realized by an abelian variety in characteristic p ? See [24], 7.11 for results and references.

(3.3) **Exercise** (Manin). Consider the curve given by $Y^2 = X^7 - X + 1$ over $\mathbb{F}_3$ (we mean the nonsingular, complete model of this affine curve). Compute the NP of its Jacobian

On could try to give an answer to construct enough curves giving all possible sNP's for their Jacobian. (We write $\mathcal{N}(C)$ instead of $\mathcal{N}(\mathrm{Jac}(C))$.) In [15] we see several examples of Jacobians in positive characteristic realizing a given NP, motivating Manin to pose the conjecture. However this attempt failed up to now because the following problem is wide open:

(3.4) **Question / open problem.** Which NP's do appear on the moduli space of curves?

(Experience: such questions are easier to answer on $\mathcal{A}_g \otimes \mathbb{F}_p$ than on $\mathcal{M}_g \otimes \mathbb{F}_p$.)

As for now we formulate:

(3.5) **Conjecture.** Suppose given g_1 and g_2 and symmetric Newton polygons ξ_1, ξ_2 for these values. Suppose that ξ_1 does appear on $\mathcal{M}_{g_1} \otimes \mathbb{F}_p$ and ξ_2 does appear on $\mathcal{M}_{g_2} \otimes \mathbb{F}_p$. We conjecture that in this case $\xi_1 \cup \xi_2$ does appear on $\mathcal{M}_{g_1+g_2} \otimes \mathbb{F}_p$. (Here $\xi_1 \cup \xi_2$ stands for the NP having the union of the set of slopes of x_1 and ξ_2, ordered in non-decreasing order.) See [23], 8.5.7.

(3.6) A corollary of this conjecture would be that for every p and for every g there exists a supersingular curve of genus g. This is know in special cases, and this is known (Van der Geer - Van der Vlugt) for $p = 2$ and all g.

(3.7) Question. It seems that questions which NP do or do not appear on $\mathcal{M}_g \otimes \mathbb{F}_p$ depends on whether we want to answer this for non-hyperelliptic or for hyper elliptic curves. It might very well be that the answers which NP appear on the moduli space of curves are different in these to separate cases; examples are scarce, and it seems no general theory does exist. Here is one example (in one of the most simple cases).
Example. Choose $p = 2$ and $g = 3$ and $\sigma = 3{\cdot}(1,1)$ the supersingular NP. There is no supersingular (non-singular) hyperelliptic curve in this case, and there is a 2-dimensional moduli space of supersingular (non-singular) non-hyperelliptic curve.

Methods, continued: algebraizability

We have seen that deformation theory as formulated by Schlessinger generalizes classical ideas and methods. But more classical idea have to be imitated and generalized. Just think of classical theory of deforming a complex torus, is the result an abelian variety? theory by Lefschetz, Chow and many others did show us how to proceed in the classical case. Also here we see this development.

(3.8) Suppose X_0 given, and consider the pro-representability of the functor

$$\mathcal{D}_{X_0} : \Lambda_\kappa \longrightarrow \text{Sets}.$$

The result is a *formal* scheme S and a formal scheme $\mathcal{X} \to S$. Are we satisfied? in general this does not answer the quest for (deformation or) lifting to an actual algebraic scheme.

(3.9) Classical theory used an embedding into an algebraic variety, usually a projective space. On of the central theorems is the Chow-GAGA technique: *a complex analytic space, realized as a closed subset of a projective space is algebraizable* (i.e. is the analytic space underlying an algebraic variety). See [26].

(3.10) The Chow-GAGA-Grothendieck theorem. In [6], III^1.5.4 we find the generalization in the language of schemes; this result (roughly) says that a formal scheme carrying a very ample divisor is algebraizable.

Here are some consequences.

The universal family of the formal deformation of an algebraic curve C_0 is algebraizable.

The universal family of the formal deformation of a polarized abelian variety is algebraizable.

Serre-Tate theory

(3.11) Let $A \to S$ be an abelian scheme over an arbitrary base. Let p be a prime number. Consider $X = A[p^\infty]$. This is a p-divisible group over S. In many cases X reflects p-adic properties of the abelian variety A.
Philosophical remark. The objects feels as an arithmetic object, rather than an object in algebraic geometry.
Exercise. Give a p-divisible group over a field K with p invertible in K, but such that there is no finite extension $K \subset K'$ such that there is no isomorphism $X \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h$ to a "constant" p-divisible group.

Exercise. Give two p-divisible groups X, Y over a discrete valuation ring, and a homomorphism $\varphi : X \to Y$ such that on the generic fibers we have an isomorphism $\varphi_\eta : X_\eta \xrightarrow{\sim} Y_\eta$, but on the especial fibers $\varphi_0 : X_0 \xrightarrow{\sim} Y_0$ this is not an isomorphism.
Exercise. Let A be an ordinary abelian variety of dimension g over an algebraically closed field $k \supset \mathbb{F}_p$. Show that

$$A[p^\infty] \cong (\mu_{p^\infty})^g \times (\mathbb{Q}_p/\mathbb{Z}_p)^g \,.$$

(3.12) Theorem (Serre and Tate). *Let A_0 be an abelian variety over a* perfect *field $\kappa \supset \mathbb{F}_p$ and $X_0 = A_0[p^\infty]$. The functor*

$$(\mathcal{A}, R) \mapsto (\mathcal{X}, R) \quad \textit{with} \quad \mathcal{A} \otimes \kappa = \mathcal{X} \otimes \kappa$$

for local artin rings R with residue class field κ, is an equivalence of categories.

(3.13) As a corollary: *for an ordinary abelian variety A_0 over a finite κ as above, its p-divisible group*

$$X_0 \cong (X_0)_{\rm loc} \times (X_0)_{\rm et}$$

and any complete local ring $R \twoheadrightarrow \kappa$ admits a lift to $\mathcal{X}' \times \mathcal{X}''$; this lift by the Serre-Tate theorem results in a formal abelian scheme $\mathcal{A} \to \mathrm{Spf}(R)$. This formal abelian scheme is algebraizable. It is a CM lift of A_0.
Conclusion. For an ordinary abelian variety over a finite field a CM lift does exist.

An interesting question. Suppose C_0 is an algebraic curve in characteristic p, and suppose its Jacobian $A_0 = \mathrm{Jac}(C_0$ is ordinary. By Serre-Tate theory there exists a canonical lift A to characteristic zero; does it follow that A (perhaps after extending its field of definition) is the Jacobian of an algebraic curve in characteristic zero? Basically the answer to this question is known.

In a sense the construction of the canonical lift is an arithmetic operation:

(3.14) Exercise (Serre). Take $p = 5$. Show an elliptic curve E over $\kappa = \mathbb{F}_5$ is supersingular if and only if $j(E) = 0$. Hence the j-values $1, 2, 3, 4 \in \mathbb{F}_5$ give ordinary elliptic curves; compute for these curves the ring of endomorphisms; shows these rings have class number equal to one; determine the j-values of these four canonical lifts. (You might want to consult [29], the last lines of § 5.)
Remark. A lot of work has been done on determining the canonical lifts and their j-values of ordinary elliptic curves. In general that problem is hard, but in the particular case of $\kappa = \mathbb{F}_5$ this is easier, because wee know all class number one quadratic fields, see [28], the last lines of Section 2.

(3.15) Remark. If ℓ is invertible in a field K (or, on a base S) (and p is an arbitrary prime number) the notion of p-divisible group $-[p^\infty]$ and Tate-ℓ-group $T_\ell(-)$ are closely related (dual in a certain sense). If p is not invertible in K, please do not use the notation $T_p(A)$ (sometimes used for the "physical Tate model", confusing).

Methods, continued: a method by Mumford

Comment. As mathematicians we love general theory. For a lifting problem (or a deformation problem) in many cases there exists a general approach (which does not imply a general answer): formulate the correct moduli problem, prove is (pro-)representable and (simply ?) read off properties of the generic fiber, $\cdots$!?

In some cases this is successful, and we are happy. This general approach was called by Grothendieck either brute force or " ... immerse the nut in a softening fluid, and wait until it opens by itself". We have the great privilege that Grothendieck considered many cases, and we can be sure that if an abstract solution is immediate in a case considered by Grothendieck, "the nut will open simply by itself", Grothendieck did find it. In other cases, where "the nut does not open simply by itself" we have to do the job ourselves. For a further discussion see [32], "Did earlier thoughts inspire Grothendieck ?"

Suppose a general answer is not easy to find; in order to focus your ideas, consider liftings of (C_0, G) as before (in general an obstructed problem), deformations and liftings of a polarized abelian variety (A_0, μ) where the polarization is not principal, deformations and lifting of a higher dimensional variety, and other obstructed problems.

(3.16) A method by Mumford. Serre once wrote to me: " .... one should keep in mind that a 'trick' in year N often becomes a 'theory' in year $N+20$." Suppose you have a structure X_0 where the general deformation problem is obstructed, where general considerations do not give "automatically" the answer you are looking for. As far as I know Mumford was the first to follow the following scheme of thought:

(defo) Start by deforming X_0 from a (difficult) situation to a generic fiber X_η; often this part of the proof involves non-canonical choices, nasty (or interesting ?) computations, consideration of complicated examples; usually it is hard to perform this part of the proof before you know many examples inside-out.

(general theory) Suppose that the situation X_η obtained does admit a solution to the original problem by "general methods", then you win. In this conference we we see such an approach in detail: try to recognize the pattern. In the question of liftings of abelian varieties (where Mumford proposed this 'trick' or 'theory') we will discuss this below.

We observe the following problem. Suppose we try to lift some X_0 to a local artin ring R, suppose it is already lifted to X' over R' but then we obtain that the obstruction $o(R \to R', X') \neq 0$. How to proceed? Maybe changing X'/R' will make the obstruction to vanish? Maybe for whatever change the obstruction remains non-zero? In general, once we encounter non-zero obstructions other methods see above have to be applied first, and then in the second part of the method above general methods (existence of moduli spaces, pro-representability of a deformation functor) are needed, can be of help.

We will see this problem in many situations, and, remarkably, in different situations the solution can be different. In some of such cases a global approach, an invariant, or something like that can be of help. Here is a situation where any change of X'/R' does not work.

4 Liftings of higher dimensional varieties

We have seen (general theory) that formations / liftings of algebraic curves exist; the two ingredients were: $\mathrm{H}^2(C_0, \mathcal{T}_{C_0}) = 0$ (obstructions vanish: formal liftings are possible) and every curve admits a very ample divisor (algebraizability is possible). As soon as one of these conditions (or both) is (are) lost in a more general situation, what can we do?

(4.1) Obstructed liftings of an algebraic varieties (Serre, Illusie, Deligne). In [27] we find examples of algebraic varieties (of dimension at least two) in positive characteristic p

such that the formal deformation problem is annihilated by a power of p; in those case even a formal lifting to a characteristic zero ring is not possible.

In Chapter 8 of [9], written by Illusie, with a letter by Serre in an appendix, these examples are worked out in more detail. In [4] we see that even in low characteristic there are counter examples.

5 Deformations and liftings of abelian varieties

(5.1) Theorem (Grothendieck). *Let A_0 be an abelian variety over a field κ of dimension* dim)$A_0 = g$. *The formal deformation space of A_0 is a formal scheme, formally smooth, over κ respectively over $W = W_\infty(\kappa)$, of relative dimension g^2,*

$$\mathrm{Def}(A_0, \mu) \cong \mathrm{Spf}\,(W[[T_{i,j} \mid 1 \leq i, j \leq g]])\,.$$

Although $\mathrm{H}^2(A_0, \mathcal{T}_{A_0}) \neq 0$ Grotendieck shows that in this case obstructions vanishes; for details see [19], Lemma 2.3.2 (for $p \neq 2$ this is easy: consider the action of multiplication by -1; try to do this exercise; how would you show obstructions vanish in this case if $p = 2$?).

(5.2) Theorem (Grothendieck, Mumford). **(a)** *Suppose (A_0, μ) is a polarized abelian variety. The deformation space* $\mathrm{Def}(A_0, \mu) \subset \mathrm{Def}(A_0)$ *can be given by $g(g-1)/2$ equations.*
(b) *In case μ is a separable polarization the deformation problem* in mixed characteristics $\mathrm{Def}(A_0, \mu)$ *is formally smooth on $g(g+1)/2$ parameters,*

$$\mathrm{Def}(A_0, \mu) \cong \mathrm{Spf}\left(W[[T_{i,j} \mid 1 \leq i, j \leq g]]/(T_{i,j} - T_{j,i})\right).$$

(Note the analogy with the same problem in characteristic zero, and the Riemann symmetry conditions.) *We conclude that any principally polarized abelian variety can be lifted to characteristic zero.*

We indicate the way Grothendieck shows the deformation problem for *principally* polarized (p-divisible groups or) abelian varieties (or,more generally for separably polarized abelian varieties) is unobstructed; this comes as a surprise because for abelian varieties of dimension at least two $\mathrm{H}^2(A_0, \mathcal{T}_{A_0}) \neq 0$. Suppose we already have a lifting $\mathcal{X}'$ to R' and $R \twoheadrightarrow R'$ is a small surjection. In this quite general situation we want to show that the obstruction $o(R \to R', \mathcal{X}')$ equals zero.

Let us first treat the case $\mathrm{char}(\kappa) \neq 2$ (although we could treat all cases at the same time). Consider the homomorphism $\iota = [-1]$ on $\mathcal{X}'$. observe that

$$o(R \to R', X') \in \mathrm{H}^2(A_0, \mathcal{O}_{A_0}) \otimes \mathfrak{t}_{A_0} \otimes I \cong \left((\mathfrak{t}_{A_0})^{\mathrm{dual}} \wedge (\mathfrak{t}_{A_0})^{\mathrm{dual}}_{A_0}\right) \otimes \mathfrak{t}_{A_0} \otimes I.$$

We conclude that

$$\iota^*(o(R \to R', \mathcal{X}')) = -o(R \to R', \mathcal{X}').$$

We show that obstructions are invariant under isomorphisms, and we conclude $o(R \to R', \mathcal{X}') = 0$.

Exercise. In the general case, including the case of residue characteristic 2, use the automophism of $\mathcal{X}'$ given by $(x, y) \mapsto (x + y, y)$ and show $o(R \to R', X') = 0$. For details see [19], 2.2: local moduli for abelian varieties.

(5.3) Exercise. Show there exist abelian varieties over an algebraically closed field not allowing a principal polarization.
Question. Observe that any elliptic curve admits a principal polarization. I expect the following to be true: for any abelian variety A of dimension at least 2 over an algebraically closed field k there exists an isogeny $A \sim B$ such that B does not admit a principal polarization.

(5.4) Grothendieck was interested in the question whether any abelian variety could be lifted to characteristic zero. See [31], page 67, [16], page 704, [32], page 260. At some point (1965), probably Grothendieck had the following example in mind; we will discuss this example, which will also show up in a disguise in the CM lifting problem.

(5.5) An example: supersingular abelian surfaces. Consider $\kappa \supset \mathbb{F}_p$ and a supersingular elliptic curve E over κ. We know that $\alpha_p \subset E$; actually this characterizes supersingularity for elliptic curves; we have that this finite subgroup scheme is the kernel of $F : E \to E^{(p)}$. Moreover $\mathrm{Hom}_\kappa(\alpha_p, \alpha_p) \cong \kappa$). Fixing coordinates by fixing $\alpha_p \subset E$ we see that

$$\mathrm{Hom}_\kappa(\alpha_p, E \times E) = \mathbb{P}^1(\kappa).$$

For any $t \in \mathbb{P}^1(\kappa)$ we obtain an abelian surface

$$\alpha_p \stackrel{t}{\hookrightarrow} E \times E \twoheadrightarrow A_t := (E \times E)/t(\alpha_p).$$

At first sight this seems weird. For example for $t = [t : 1] \in \mathbb{P}^1(\mathbb{F}_p(t))$, where t is a transcendental, we obtain an abelian surface with CM, however it cannot be defined over a finite field.
Exercise. Does this A_t (with any polarization it) allow a lift to characteristic zero ?

(5.6) If you want to see an answer to the question of liftability of abelian varieties in this special case, or in the general case of lifting abelian varieties to characteristic zero, and if you want to see how Mumford's method works in this case, see [18]. In this proof, as initiated by Mumford, and written up by the authors of [18], first a deformation of an abelian variety in equi-characteristic to an ordinary abelian variety is constructed (a difficult, non-canonical construction); then the theory of Serre-Tate canonical liftings ends the construction.

From these results it follows that $\mathrm{Def}(A_0, \mu)$ is a complete intersection scheme, having relative dimension $g(g+1)/2$.

(5.7) Question. Is there an A_0 be an abelian variety over $\kappa \subset \mathbb{F}_p$ that does not allow a lifting to a normal domain?

6 Deformations of abelian varieties with given Newton polygons

Grotendieck showed that under specialization of abelian varieties in positive characteristics "NP's go up", see [8], [12], Th. 2.3.1 on page 143. Grothendieck conjectured the converse (actually for p-divisible groups, also called Barsotti-Tate groups):

(6.1) A conjecture by Grothendieck. Suppose given a p-divisible group X_0 and NP's $\zeta = \mathcal{N}(X_0)$ and τ "below" ζ (i.e. no point of τ is strictly above ζ). There does exist a deformation (in equal characteristic) of X_0 with generic fiber $\mathcal{N}(X_\eta) = \tau$. See [8], page 150.

Discusion. Here again we see the general pattern. The deformation space $\mathcal{W}_\tau(\mathrm{Def}(X_0))$, considering only NP's equal or above τ, is (pro-)representable. In order to prove this Grothendieck conjecture we want to show there is at least one fiber with NP equal to τ. Experience shows that it is hard to read off the NP over an arbitrary base. How do you proceed? It seems that "... immerse the nut in a softening fluid, but here it does not open by itself ..." (and as far as I know, Grothendieck did not come back to this question).

Then you can consider the case of a quasi-polarized p-divisible group (or of a polarized abelian variety). Does a deformation to a given NP exist? (An analog inspired by the conjecture by Grothendieck). What do you think the outcome will be?

For references to results see [3], Sections 7 and 8, see [24], Section 8.

It turned out this was a beginning of unravelling fine structures, stratifications and foliations on the moduli space of polarized abelian varieties in positive characteristic, as started by Manin, Grothendieck, Katz and many others.

7 CM liftings of abelian varieties

We say an abelian variety A of dimension g over a field K admits *sufficiently many complex multiplications* (or, A is a CM abelian variety), if $\mathrm{End}^0(A)$ contains a semi-simple commutative sub algebra of rank $2g$ over $\mathbb{Q}$. (A further discussion will be given later this week.) Tate showed that any abelian variety over a finite field is a CM abelian variety.

(7.1) Question. Does every abelian variety A_0 defined over a finite field admit a CM lifting (to characteristic zero) ?

(7.2) An easy counterexample. One can ask whether any CM abelian variety in positive characteristic does allow a CM lifting. Let t be a transcendental over $\mathbb{F}_p$ and left A_0 given as $A_0 = (E \times E)/t(\alpha_p)$ as before. We see that A_0 cannot be defined over a finite field. However any CM abelian variety in characteristic zero is defined over a number field. The existence of a CM lift of this A_0 would give a contradiction.

This simple-minded example does not answer the CM lifting question directly. However in 1992 in the paper [21] we see that the examples inspires to give infinitely many counterexamples tot the lifting problem (actually for any $g > 2$). This again is an example of an obstructed lifting problem, where a general approach does not give a satisfactory answer. Later in the week I will further discuss this lifting problem and tell you the story of the complete answer that can be given to CM lifting questions.

(7.3) Exercise. Suppose E is a supersingular elliptic curve over $\mathbb{F}_2$. Let $F : E \to E^{(p)} \cong E$ be its Frobenius morphism. Does there exist a lifting of (E, F) to an unramified domain ?

(7.4) A miracle. We have seen a *difficult problem*: algebraizability of formal liftings. We have seen in the Serre-Tate theory of lifting ordinary abelian varieties there is automatic. More generally: a CM lift, a priori being formal, is algebraizable under the condition that the CM algebra acts via a CM Type, see [2], Theorem 2.2.3.

(7.5) An obvious remark. Liftability of an abelian variety is proved by first deforming to an ordinary abelian variety. Of course, this idea fails for the CM case, because a "CM deformation" does not perform what you would require.

Conclusion. We have seen:

- Lifting algebraic curves and principally polarized abelian varieties is possible, as follows from general theory.
- (We will see this week what the answer is to the question of lifting an algebraic curve with some group of automorphisms ... remember the Mumford approach.)
- Lifting higher dimensional algebraic varieties in general is not possible.
- Lifting abelian varieties,

  CM lifting of abelian varieties after an appropriate isogeny,

  and deforming principally polarized abelian varieties to a given NP

  are possible,

  but proofs are involved, and do not follow from general theory only.

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Here are three inspiring books you will love to read:

[31] *Correspondance Grothendieck-Serre.* Edited by Pierre Colmez and Jean-Pierre Serre. Documents Mathématiques (Paris), **2**. Soc. Math. France, Paris, 2001.

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[34] (An edition of letters between Jean-Pierre Serre and John Tate.) [To appear.]

Appendix. Text of an email 9-VIII-2014 by Deligne (reproduced here with his permission) to P. Blass.
The argument of Serre showing that in char. p no integral cohomology can exist can be used to exhibit a smooth surface S in char. 2 (or 3) such that no smooth surface S' mapping to S, whose function field is a purely inseparable extension of $k(S)$, can be lifted to char. 0.

The criterion used is the following. Suppose S has an etale Galois covering S_1 with Galois group G and that the ℓ-adic H^1 of S_1, as a representation of G, has no $\mathbb{Q}$-form. The same then holds for S' : S_1 pulls back to an etale Galois covering S_1' of S', and S_1 and S_1' have the same H^1. For a liftable surface, to have such a covering is impossible: if the surface lifts, the covering lifts too, the H^1 of the lifting of the covering would be the previous H^1, with the same action of G, but now a Betti cohomology gives a $\mathbb{Q}$-form.

It remains to find such a S. Start with the supersingular elliptic curve E of char. 2 (or 3). Its automorphism group G is of order 24 (the units of a maximal order the obvious $\mathbb{Q}$-form of the quaternions) (or ...). The action of G on $H^1(E)$ has no $\mathbb{Q}$-form. Take a curve C on which G acts freely. The example is $S := (E \times C)/G$, with covering $E \times C$. The representation of G on $H^1(E \times C) = H^1(E) \times H^1(C)$ has no $\mathbb{Q}$-form. Indeed, for a representation with a rational character, having a $\mathbb{Q}$-form means that some (complex) irreducible components come with a multiplicity divisible by some integer, and one uses that the representation on $H^1(C)$ has a $\mathbb{Q}$-form : it is a multiple of the regular plus two copies of the trivial representation (by the Lefschetz trace formula).

To find C : start with the regular representation of G, call it R, and the projective space P at infinity. G acts on P with a unique fixed point. Take a general iterated hyperplane section of dimension one of P/G, and $C :=$ its inverse image in P. Best, Pierre Deligne

Frans Oort
Mathematical Institute
Pincetonplein 5
NL - 3584 CC Utrecht
The Netherlands
email: f.oort@uu.nl