Regularization of a sharp shock by the defocusing nonlinear Schrödinger equation

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Statement of the Problem

We study the defocusing NLS equation in $\mathbb{R} \times [0, \infty)$:

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - |\psi|^2\psi = 0,$$
$$\psi(x,0) = \psi_0(x) = A(x)e^{iS(x)/\epsilon}$$

- Here ϵ is the dispersion parameter.
- Small dispersion limit: study solution $\psi(x,t;\epsilon)$ as $\epsilon \to 0$ for (x,t) fixed (bounded)
- Long-time limit: study solution $\psi(x,t;\epsilon)$ for ϵ fixed as $t \to \infty$.

Hydrodynamic form

The change of variables

$$\rho = |\psi|^2 \qquad \text{and} \qquad u = \epsilon \, \mathrm{Im}(\psi_x/\psi)$$

transforms the NLS equation into the system of hydrodynamic conservation laws

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + \left(\rho u^2 + \frac{1}{2}\rho^2 \right)_x = \frac{\epsilon^2}{4} \left(\rho (\log \rho)_{xx} \right)_x$$

When $\epsilon = 0$ these are the Euler equations for an ideal compressible gas with density ρ, velocity u, and pressure $P = \rho^2/2$.

Shock formation

- For smooth initial data the r.h.s can be formally neglected in the zero dispersion limit.

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + \left(\rho u^2 + \frac{1}{2}\rho^2 \right)_x \approx 0$$

- Euler equations can be solved by method of characteristics.
- Solutions of the Euler equations can develop gradient catastrophes (infinite derivatives)

Shock formation

Once large gradients are developed, the r.h.s. is not perturbative.

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + \left(\rho u^2 + \frac{1}{2}\rho^2\right)_x = \frac{\epsilon^2}{4}\left(\rho(\log \rho)_{xx}\right)_x$$

For $\epsilon > 0$, shock formation is regularized by emerging regions of

- Rarefaction waves
- Slowly modulating $\mathcal{O}(\epsilon)$ wavelength oscillations with $\mathcal{O}(1)$ amplitudes: dispersive shock waves (DSWs).

Modulation Theory

- NLS admits exact G-phase wave solutions for any $G \in \mathbb{Z}_+$, described by $G+2$ Riemann invariants.
- Rapidly oscillatory solutions of small dispersion NLS are locally well described by slowly modulating multi-phase wave solutions of NLS.
- The slow evolution of the Riemann-invariants can be computed by averaging (Whitham theory).
- The Riemann invariants $\lambda_j = \lambda_j(x,t)$ satisfy a hyperbolic system of equations

$$\partial_t \lambda_j + v_j(\boldsymbol{\lambda}) \partial_x \lambda_j = 0, \quad j = 1, 2, \ldots, 2G+2$$

Zero phase modulation

Before breaking Euler equations are valid and solution resembles a slowly modulated plane wave

$$\psi(x,t) \approx \sqrt{\omega - k^2} e^{iS/\epsilon}, \quad \partial_x S = k, \quad \partial_t S = \omega$$

The Euler equations are diagonalized by the Riemann-invariants $\lambda_{1,2} = -u/2 \pm \sqrt{\rho}$.

$$\partial_t \lambda_j + v_j(\boldsymbol{\lambda}) \partial_x \lambda_j = 0,$$
$$v_1(\boldsymbol{\lambda}) = -\frac{1}{2}(3\lambda_1 + \lambda_2), \qquad v_2(\boldsymbol{\lambda}) = -\frac{1}{2}(\lambda_1 + 3\lambda_2)$$

One phase modulation

Single phase waves are described by four invariants:
$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$

$$\rho = a_1^2 - (a_1^2 - a_3^2)\,\mathrm{dn}^2\left(\sqrt{a_1^2 - a_3^2}\,\frac{x - Vt}{\epsilon}, m\right)$$

$$\rho u = \rho V - a_1 a_2 a_3$$

where

$$a_1 = -(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)/2$$
$$a_2 = -(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)/2$$
$$a_3 = -(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)/2$$

$$V = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2$$
$$m = \frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}$$

One phase modulation

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$$\rho = a_1^2 - (a_1^2 - a_3^2)\,\mathrm{dn}^2\left(\sqrt{a_1^2 - a_3^2}\,\frac{x - Vt}{\epsilon}, m\right)$$

$$\rho u = \rho V - a_1 a_2 a_3$$

where $\partial_t \lambda_j + v_j(\boldsymbol{\lambda})\partial_x \lambda_j = 0$ with

$$v_j(\boldsymbol{\lambda}) = V + \left(2\frac{\partial}{\partial \lambda_j}\log L(\boldsymbol{\lambda})\right)^{-1}$$

$$L(\boldsymbol{\lambda}) = \sqrt{2}\int_{\lambda_2}^{\lambda_1} \frac{d\tau}{\sqrt{-\prod_{j=1}^{4}(\tau - \lambda_j)}}$$

Sharp shock initial data

Our goal is to understand the evolution of the following class of initial data:

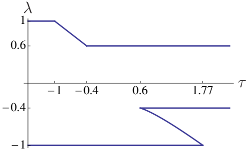
$$\psi_0(x) = \begin{cases} 1 & x < 0 \\ Ae^{-2i\mu x/\epsilon} & x > 0. \end{cases}$$

for real constants μ and $A > 0$.

Each half-line represents a plane (0-phase) wave with Riemann invariants ± 1 on the left side and $\lambda_\pm = \mu \pm A$ on the right.

Whitham theory for the sharp shock

The modulation theory for exactly this problem was worked out by El, Gurevich, Geogjaev, and Krylov *Phys D.* 87 (1995):



- Scale free data: $\Rightarrow$ λ_j's evolution is self-similar: $\lambda = \lambda(x/t)$.
- $(v_j(\boldsymbol{\lambda}) - \tau)\frac{\partial \lambda_j}{\partial \tau} = 0$:
  - λ_j constant or $v_j(\boldsymbol{\lambda}) = \tau$.
  - Strict hyperbolicity: $v_j < v_k$ for $\lambda_j < \lambda_k$ implies only one λ_j non-const.
- For $x \gg 1$ $(x \ll -1)$ solution is a plane wave with Riemann invariants matching the initial data: $\lambda_{1,2} = \lambda_{\pm}\ (\pm 1)$.

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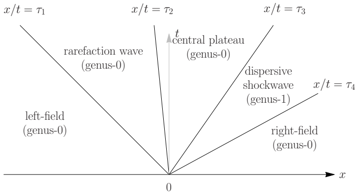


- Shock is regularized by five zones emerging from the origin: left, center, and right constant states (plane waves) connected by rarefactions or DSWs
- Type of connections depend on the relative ordering of right invariants $\lambda_{\pm} = \mu \pm A$ with respect to left invariants ± 1.

What's left to do?

We have three interrelated goals:

- Rigorous justification: can the predictions of Whitham theory be recovered from a direct solution method.
- Phases: can we recover the slowly evolving terms in complex phase of ψ lost in the averaging theory?
- Error estimation: for small finite ϵ (or large fixed t) how close is the true solution to the asymptotic approximation.

The inverse scattering procedure allows us to answer all of these questions.

Lax Pair

The NLS equation is integrable in the sense that it has a Lax-Pair representation:

$$\epsilon v_x = \mathcal{L} v \qquad i\epsilon v_t = \mathcal{B} v$$

with

$$\mathcal{L} = \mathcal{L}(z; x, t) = -iz\sigma_3 + \Psi$$
$$\mathcal{B} = \mathcal{B}(z; x, t) = iz\mathcal{L} + \frac{1}{2}(\Psi^2 + \epsilon\Psi_x)\sigma_3$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Psi = \begin{pmatrix} 0 & \psi(x,t) \\ -\psi(x,t)^* & 0 \end{pmatrix}$$

The NLS equation is equivalent to the zero-curvature condition:

$$i\epsilon\mathcal{L}_t + \epsilon\mathcal{B}_x + [\mathcal{L}, B] = 0$$

Forward scattering

The *forward scattering step* involves studying the spectrum of the Lax operator $\epsilon v_x = (-iz\sigma_3 + \Psi)v$.
One must compute:

1. A reflection coefficient $r(z)$ defined on the continuous spectrum of the operator
2. Any discrete eigenvalues of the operator and associated normalization coefficients: $\{z_k, c_k\}_{k=1}^N$
3. Once this scattering data is computed from initial data, the time evolution is trivial.

Forward scattering: step-like data

For steplike initial data the continuous spectrum is not the whole real axis, but instead has four branch points at each of the four asymptotic Riemann invariants: $-1, 1, \lambda_-, \lambda_+$

- For our piecewise constant data, everything is computed explicitly without introducing Riemann surfaces.

$$\beta_L(z) = \left(\frac{z-1}{z+1}\right)^{1/4} \qquad \beta_R(z) = \left(\frac{z-\lambda_+}{z-\lambda_-}\right)^{1/4}$$

$$a(z) = \frac{\beta_L(z)\beta_R(z)^{-1} + \beta_L(z)^{-1}\beta_R(z)}{2}, \quad b(z) = \frac{\beta_L(z)\beta_R(z)^{-1} - \beta_L(z)^{-1}\beta_R(z)}{2i},$$

$$r(z) = \frac{b(z)}{a(z)} = -i\frac{\beta_L(z)^2 - \beta_R(z)^2}{\beta_L(z)^2 + \beta_R(z)^2}.$$

- $r(z)$ is analytic in $\mathbb{C}\backslash((-1,1)\triangle(\lambda_-,\lambda_+))$ and
- For $z \in (-1,1)\triangle(\lambda_-,\lambda_+)$, $|r_\pm(z)| = 1$

Inverse scattering

For each (x,t) find $\mathbf{M}(z;x,t) : \mathbb{C}\backslash\mathbb{R} \to SL(2,\mathbb{C})$ such that

1. $\mathbf{M}$ is analytic in $\mathbb{C}\backslash\mathbb{R}$, where $\mathbb{R}$ is oriented left-to-right.
2. $\mathbf{M}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\mathbf{M}(z)$ is bounded at each finite z except the points p, $p \in \{\lambda_-, \lambda_+\}$ where it admits the singular behavior

$$\mathbf{M}(z) = \mathcal{O}\begin{pmatrix} (z-p)^{1/4} & (z-p)^{-1/4} \\ (z-p)^{1/4} & (z-p)^{-1/4} \end{pmatrix}, \quad z \in \mathbb{C}^+, \qquad p \in \{\lambda_-, \lambda_+\},$$

$$\mathbf{M}(z) = \mathcal{O}\begin{pmatrix} (z-p)^{-1/4} & (z-p)^{1/4} \\ (z-p)^{-1/4} & (z-p)^{1/4} \end{pmatrix}, \quad z \in \mathbb{C}^-, \qquad p \in \{\lambda_-, \lambda_+\}.$$

4. For $z \in \mathbb{R}$, m satisfies the jump relation $\mathbf{M}_+(z) = \mathbf{M}_-(z)v(z;x,t)$

Once $\mathbf{M}$ is found, the solution of the NLS initial value problem is given by

$$\psi(x,t) = 2i \lim_{z\to\infty} z\mathbf{M}_{12}(z)$$

The jump matrix

All of the dependence of the problem on initial data and (x,t) is encoded in the jump matrix

$$v(z,x,t) = \begin{cases} \begin{pmatrix} 1 - rr^* & -r^* e^{-2i\theta/\epsilon} \\ r e^{2i\theta/\epsilon} & 1 \end{pmatrix} & z \in \mathbb{R}\backslash(I_L \cup I_R) \\ \begin{pmatrix} 0 & -r_-^* e^{-2i\theta/\epsilon} \\ r_+ e^{2i\theta/\epsilon} & 1 \end{pmatrix} & z \in I_L\backslash(I_L \cap I_R) \\ \begin{pmatrix} (a_+ a_-^*)^{-1} & -e^{-2i\theta/\epsilon} \\ e^{2i\theta/\epsilon} & 0 \end{pmatrix} & z \in I_R\backslash(I_L \cap I_R) \\ \begin{pmatrix} 0 & -e^{-i\theta/\epsilon} \\ e^{i\theta/\epsilon} & 0 \end{pmatrix} & z \in I_L \cap I_R \end{cases}$$

Here $I_L = (-1,1)$ and $I_R = (\lambda_-, \lambda_+)$ and $\theta = tz^2 + xz$.

Controlling rapid oscillations

The Deift-Zhou steepest descent method for Riemann-Hilbert problems has several steps, the first and often most difficult, is to control the rapid oscillations in the jump matrices. This is accomplished by the transformation $\mathbf{M}(z) = \mathbf{N}(z)e^{ig(z)\sigma_3/\epsilon}$, where g is analytic in $\mathbb{C}\backslash(I_L \cup I_R)$ and $g(\infty) = 0$. The new RHP for n is

1. $\mathbf{N}$ is analytic in $\mathbb{C}\backslash\mathbb{R}$, where $\mathbb{R}$ is oriented left-to-right.
2. $\mathbf{N}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\mathbf{N}(z)$ is bounded at each finite z except the points p, $p \in \{\lambda_-, \lambda_+\}$ where it admits the same singular behavior.
4. For $z \in \mathbb{R}$, n satisfies the jump relation $\mathbf{N}_+(z; x, t) = \mathbf{N}_-(z; x, t) v_{\mathbf{N}}(z; x, t)$

$$v_{\mathbf{N}} = \begin{bmatrix} \#_{11} e^{2\Delta(z)/\epsilon} & \#_{12} e^{-2ih(z)/\epsilon} \\ \#_{21} e^{2ih(z)/\epsilon} & \#_{22} e^{-2\Delta(z)/\epsilon} \end{bmatrix}$$

$$2h(z) = \theta(z) - g_+(z) - g_-(z) \qquad 2\Delta(z) = -i(g_+(z) - g_-(z))$$

bands and gaps

The intervals I_L and I_R are partitioned into:

- **Bands**: on which $h'(z) = 0$ with

$$\Delta(z) > 0 \text{ for } z \in I_L \backslash (I_L \cap I_R)$$
$$\Delta(z) < 0 \text{ for } z \in I_R \backslash (I_L \cap I_R).$$

- **Gaps**: on which $\Delta = 0$ and $h'(z) > 0$.

If we let $J_k = (\lambda_{2k}, \lambda_{2k-1})$ denote the k^{th} band interval, then g can be expressed in terms of a certain meromorphic differential defined on a Riemann-surface associated to the function $R(z, \boldsymbol{\lambda}) = \sqrt{\prod_{k=1}^{2G+2} (z - \lambda_k)}$.

$$g(z) = \int_{\infty}^{z} d\theta - d\varphi$$

g-function in terms of differentials

$$g(z) = \int_{\infty}^{z} d\theta - d\varphi$$

$$d\varphi = 2t\omega^{(1)} + x\omega^{(0)} = \frac{2tP_1(z;\boldsymbol{\lambda}) + xP_0(z;\boldsymbol{\lambda})}{R(z,\boldsymbol{\lambda})}dz$$

Here $\omega^{(k)}$ are normalized differentials of second kind

$$\omega^{(k)} = \pm\left[z^k + \mathcal{O}\left(z^{-2}\right)\right]dz \quad P \to (\infty, \pm\infty)$$

$$\int_{\lambda_{2j}}^{\lambda_{2j-1}} \omega^{(k)} = 0, \quad j = 1, \ldots, G$$

How are the endpoints determined?

Endpoints λ_k are split into two classes:

- **Soft edges:** λ_k varies according to the rule that

$$d\varphi/dz\bigg|_{z=\lambda_k} = 0 \quad \Rightarrow \quad x - v_k(\boldsymbol{\lambda})t = 0,\ v_k(\boldsymbol{\lambda}) = -2\frac{P_1(\lambda_k,\boldsymbol{\lambda})}{P_0(\lambda_k,\boldsymbol{\lambda})}$$

- **Hard edges:** λ_k is known and constant.

$\star$ These evolution rules exactly recapitulate the self-similar evolution of the Riemann invariants under the Whitham equations for modulations of defocusing NLS.

$$(v_k(\boldsymbol{\lambda}) - \tau)\frac{\partial \lambda_k}{\partial \tau} = 0, \quad \tau = x/t$$

Mechanism for changes of band-gap structure

The evolution of the endpoints change when the inequality conditions on the g-function:

- **Bands**: on which $h'(z) = 0$ with

$$\Delta(z) > 0 \text{ for } z \in I_L \backslash (I_L \cap I_R)$$
$$\Delta(z) < 0 \text{ for } z \in I_R \backslash (I_L \cap I_R).$$

- **Gaps**: on which $\Delta = 0$ and $h'(z) > 0$.

Failure of the inequalities corresponds to zeros of $d\varphi/dz$ entering intervals $I_L \backslash (I_L \cap I_R)$ or $I_R \backslash (I_L \cap I_R)$.

Sweeping things under the rug...

Once the oscillations are controlled. One needs to:

1. Deform onto steepest descent contours $\mathbf{P} = \mathbf{NL}$: use factorizations of the jump matrix to move oscillatory factors onto contours on which they are decaying.
2. Construct a global model $\mathbf{Q}$: as $\epsilon \to 0$, there is some limiting problem away from stationary points. Construct a global model (patched together outer and local models) which give a uniform approximation
3. Estimate the error: the error $\mathbf{E} = \mathbf{P}\mathbf{Q}^{-1}$ satisfies a small-norm RHP with a unique solution whose asymptotic expansion can be computed explicitly.

The results in a particular case

To make things completely concrete, restrict the initial data such that the right and left asymptotic Riemann invariants satisfy

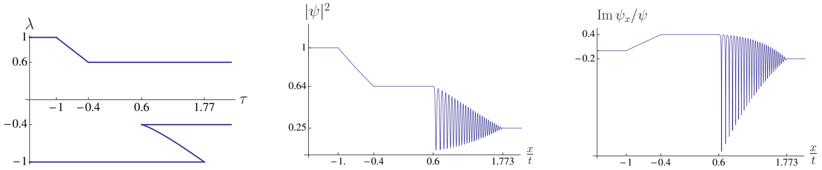
$$-1 < \lambda_- < \lambda_+ < 1 \qquad \lambda_\pm = \mu \pm A$$

Theorem

Given step initial data wtih $\lambda_\pm = \mu \pm A$ satisfing $-1 < \lambda_- < \lambda_+ < 1$, the long-time/small-dispersion asymptotic behavior of the solution $\psi(x,t)$ of the NLS equation is given by one of the five following formulae depending on the value of the similarity variable $\tau = x/t$ relative to the transition speeds τ_j identified as:

1. $\tau_1 = -1$
2. $\tau_2 = -\frac{1}{2}(-1 + 3\lambda_+)$
3. $\tau_3 = -\frac{1}{2}(-1 + 2\lambda_- + \lambda_+)$
4. $\tau_4 = -\frac{1}{2}(\lambda_+ + \lambda_- - 2) + \frac{2(1+\lambda_-)(1+\lambda_+)}{\lambda_+ + \lambda_- + 2}$

Formulae: phases and error



1. For $\tau < \tau_1$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the left half of the initial data:

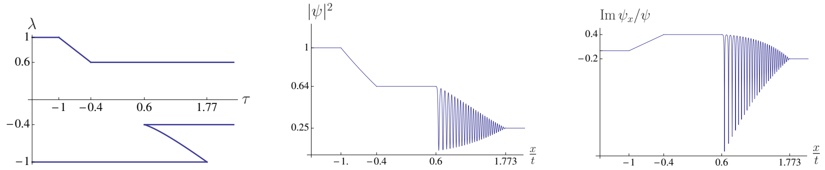
$$\psi(x,t) = e^{-it/\epsilon} e^{-i\phi(x/t)} + \mathcal{O}\left(\sqrt{\frac{\epsilon}{t}}\right) \tag{1}$$

$$\phi(\tau) = \frac{1}{\pi}\left(\int_{-\infty}^{-1} + \int_{1}^{\xi_+(\tau)}\right) \frac{\log(1-|r(z)|^2)}{\sqrt{z^2-1}}\,dz + \frac{1}{\pi}\left(\int_{-1}^{\lambda_-} + \int_{\lambda_+}^{1}\right)\frac{\arg(r_+(z))}{\sqrt{1-z^2}}\,dz$$

$$\xi_+(\tau) = \frac{1}{4}\left[\sqrt{\tau^2+8} - \tau\right]$$

or $\tau > \tau_4$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the right half of the initial data:

Formulae: phases and error



2. For $\tau_1 < \tau < \tau_2$, the solution is described by the rarefaction

$$\psi(x,t) = \left(\frac{2t-x}{3t}\right) e^{(-i/3\epsilon)(2t-2x-x^2/t)} e^{-i\phi(x/t)} + \mathcal{O}\left(\frac{\epsilon}{t}\right) \tag{1}$$

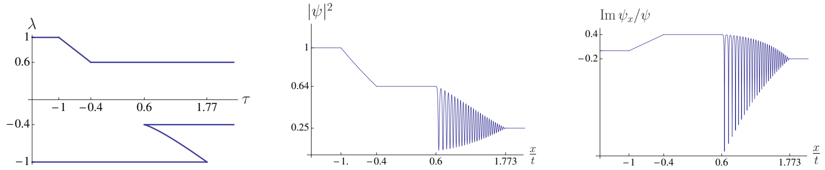
$$\phi(\tau) = \frac{1}{\pi}\left(\int_{-\infty}^{-1} \frac{\log(1-|r(z)|^2)}{\sqrt{(z+1)(z-\lambda_s(\tau))}} dz + \left(\int_{-1}^{\lambda_-} + \int_{\lambda_+}^{\lambda_s(\tau)}\right) \frac{\arg(r_+(z))}{\sqrt{(z+1)(\lambda_s(\tau)-z)}} dz\right)$$

$$\lambda_s(\tau) = \frac{(1-2\tau)}{3}$$

or $\tau > \tau_4$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the right half of the initial data:

$$\psi(x,t) = Ae^{-i(2\mu x+(A^2+2\mu^2)t)/\epsilon} e^{-i\phi(x/t)} + \mathcal{O}\left(\sqrt{\frac{\epsilon}{t}}\right)$$

Formulae: phases and error



3. For $\tau_2 < \tau < \tau_3$ the solution is asymptotically described by the (unmodulated) plane wave

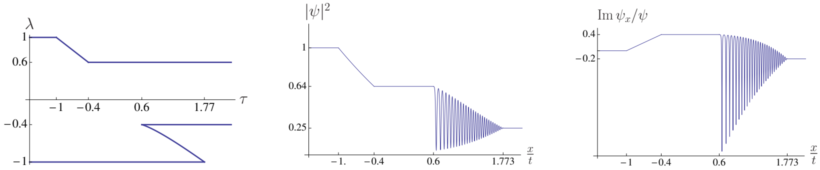
$$\psi(x,t) = \sqrt{\rho} e^{i(kx-\omega t)/\epsilon} e^{-i\phi_0} + \mathcal{O}\left(e^{-ct/\epsilon}\right) \tag{1}$$

$$\rho = \left(\frac{\lambda_+ + 1}{2}\right)^2 \qquad k = -(\lambda_+ - 1) \qquad \omega = \frac{1}{2}k^2 + \rho$$

$$\phi_0 = \frac{1}{\pi}\left(\int_{-\infty}^{-1} \frac{\log(1-|r(z)|^2)}{\sqrt{(z+1)(z-\lambda_+)}} dz + \int_{-1}^{\lambda_-} \frac{\arg(r_+(z))}{\sqrt{(\lambda_+ - z)(z+1)}} dz\right).$$

Note that, unlike the other four intervals, here the error bound is exponentially, not algebraically, small in ϵ/t.
or $\tau > \tau_4$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the right half of the initial data:

Formulae: phases and error



4. For $\tau_3 < \tau < \tau_4$ the asymptotic behavior of the solution is described by a slowly modulated one-phase (elliptic) wave, a dispersive shock wave,

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{iS(x,t)} + \mathcal{O}\left(\frac{\epsilon}{t}\right),$$

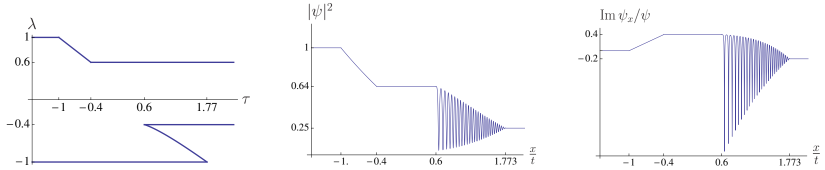
$$\rho(x,t) = a_1^2 - (a_1^2 - a_3^2)\,\mathrm{dn}^2\left(\sqrt{a_1^2 - a_3^2}\left(\frac{x - Vt}{\epsilon} + \phi\left(\frac{x}{t}\right)\right) - K(m), m\right)$$

$$S(x,t) = \frac{Vx - (a_1^2 + a_2^2 + a_3^2 - V^2)t + (x - 2Vt)\eta}{\epsilon} + 2(V+\eta)\phi\left(\frac{x}{t}\right)$$

$$+ \arg\left\{\theta_3\left[\frac{\pi}{2K(m)}\sqrt{a_1^2 - a_3^2}\left(\frac{x - Vt}{\epsilon} + \phi\left(\frac{x}{t}\right)\right) - i\pi\frac{F(\varphi, 1-m)}{K(m)}\right]\right\}$$

$$u(x,t) = \epsilon\frac{\partial S}{\partial x} = \frac{a_1 a_2 a_3}{\rho(x,t)} + V + \mathcal{O}(\epsilon/t)$$

Formulae: phases and error



4. (cont.)

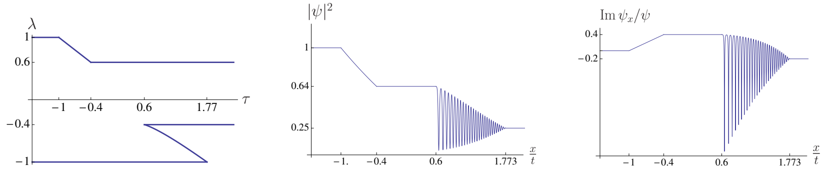
$$\phi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{-1} \frac{(z+V)\log(1-|r(z)|^2)}{\sqrt{\prod_{j=1}^{4}(z-\lambda_j)}}\,dz - \frac{1}{2\pi}\int_{-1}^{\lambda_s(\tau)} \frac{(z+V)\arg r_+(z)}{\sqrt{-\prod_{j=1}^{4}(z-\lambda_j)}}\,dz,$$

$$\eta = \lambda_1 - (\lambda_1-\lambda_4) Z\left(-\frac{\lambda_3-\lambda_4}{\lambda_1-\lambda_4}, n\right),$$

$$n = -\left(\frac{\lambda_3-\lambda_4}{\lambda_1-\lambda_4}\right), \qquad \varphi = \arcsin\sqrt{\frac{\lambda_2-\lambda_4}{\lambda_1-\lambda_4}}.$$

Here K and E are the complete elliptic integrals of the first and second kind respectively, F is the incomplete elliptic integral of the first kind, and Z is the Jacobi zeta function. or $\tau > \tau_4$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the right half of the initial data:

Formulae: phases and error



For $\tau > \tau_4$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the right half of the initial data:

$$\psi(x,t) = Ae^{-i(2\mu x + (A^2+2\mu^2)t)/\epsilon} e^{-i\phi(x/t)} + \mathcal{O}\left(\sqrt{\frac{\epsilon}{t}}\right)$$

$$\phi(\tau) = \frac{1}{\pi}\int_{-\infty}^{\xi_-(\tau)} \frac{\log(1-|r(z)|^2)}{\sqrt{(z-\lambda_+)(z-\lambda_-)}}dz,$$

$$\xi_-(\tau) = \frac{2\mu - \tau}{4} - \frac{\sqrt{(2\mu+\tau)^2 + 8A^2}}{4}.$$

5.

THE END.