Dispersive Hydrodynamics: the Mathematics of Dispersive Shock Waves and Applications

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1 Overview of the Field

Dispersive hydrodynamics is the domain of applied mathematics and physics concerned with fluid motion in which internal friction, e.g., viscosity, is negligible relative to wave dispersion. In conservative media such as superfluids, optical materials, and water waves, nonlinearity has the tendency to engender wavebreaking that is mitigated by dispersion. The mathematical framework for such media can often be described by hyperbolic systems of partial differential equations with conservative, dispersive corrections that play a fundamental role in the dynamics. Generically, the result of nonlinearity and dispersion is a multiscale, unsteady, coherent wave structure called a dispersive shock wave or DSW. Over long time scales, multiple wavebreaking events or, in focusing media, modulational instability can lead to the development of soliton (strong) turbulence. This meeting brought together an international collection of mathematicians and physicists in order to identify common interests and emerging problems involving DSWs, soliton turbulence, and their mathematical description.

This field of research has origins in soliton theory, conservation laws, and fluid dynamics. In 1965, the seminal computational work of Zabusky and Kruskal [1] demonstrated the existence of soliton solutions to the Korteweg-de Vries (KdV) equation through a process of nonlinear wavebreaking. That same year, Whitham introduced a general asymptotic approach to study modulated periodic nonlinear dispersive waves [2, 3]. Both of these works considered conservative, nonlinear, dispersive wave problems. Again in 1965, although within the context of a different field of research, Glimm's fundamental work on hyperbolic conservation laws [4] contributed to the rapid growth in understanding of this field, see, e.g. [5]. The marriage of dispersive nonlinear waves and hyperbolic conservation laws in the context of dispersive hydrodynamics was initiated in 1974 at the hands of Gurevich and Pitaevskii [6] through their study of a Riemann problem regularized by dispersion in the KdV equation. The resulting dispersive shock waves were understood utilizing Whitham's modulation equations, later shown to describe the weak, zero dispersion limit of the KdV equation by Lax, Levermore, and Venakides [8, 9]. The KdV Whitham modulation equations are now known to be strictly hyperbolic and genuinely nonlinear [10], highlighting the deep connections between conservation laws and dispersive nonlinear waves in dispersive hydrodynamics. Another fundamental work was on averaging of multiphase solutions to the KdV equation utilizing finite gap theory [7].

An essential aspect of dispersive hydrodynamics is its physical realization. Laboratory measurements of DSWs were undertaken in the context of undular bores in shallow water waves by Favre in 1935 [11].

Applications to collisionless plasma in the 1960s [12, 13] motivated the early theoretical works [1, 6]. Additional geophysical applications include internal waves in the ocean and atmosphere (e.g., the Morning Glory). More recently, experiments in ultracold atoms [14, 15] and nonlinear photonics [16, 17] have inspired further mathematical study of dispersive hydrodynamics.

The unique challenges presented by nonlinear wavebreaking and dispersion have since been explored in a variety of ways. Methods include the Inverse Scattering Transform, finite gap theory, matched asymptotic expansions, Whitham modulation theory, PDE analysis, numerical simulation, and experiment. Fifty years after the seminal year 1965, the Dispersive Hydrodynamics workshop held at BIRS May 17-22 brought together practitioners from the dispersive waves, hyperbolic conservation laws, and experimental communities with the aim of addressing recent physical and mathematical developments in the field and to identify open problems, which are now described.

2 Recent Developments and Open Problems

There are many fundamental mathematical problems emerging in the study of multiscale, dispersive hydrodynamic phenomena. Some of these are listed below.

1. Analytical and numerical description of multidimensional DSWs in KP or other integrable and nonintegrable equations. An extension of the existing theory for one-dimensional DSWs to two spatial dimensions is a long-standing problem posing a number of mathematical challenges. Despite recent progress in the understanding of pre-breaking KP dynamics as well as careful numerical simulations of some multidimensional dispersive regularization problems, breakthroughs in this area still lie ahead. A closely related direction which has been under active development in recent years is the theory of integrable hydrodynamic type systems in higher dimensions.

2. Modulation theory for hydrodynamic systems with non-strict hyperbolicity/linear degeneracy regularized by dispersive terms. The majority of existing DSW studies are related to systems which can be characterized as genuinely nonlinear, strictly hyperbolic conservation laws modified by small dispersion terms. The DSWs generated in such systems represent dispersive counterparts of classical, Lax shocks. At the same time, there exists a rich mathematical theory of nonclassical shock waves in hydrodynamic systems lacking genuine nonlinearity and/or strict hyperbolicity. The study of purely dispersive counterparts of nonclassical, dissipative shock resolutions is one of the outstanding problems in dispersive hydrodynamics. There have been several recent important works in this direction but a unified mathematical framework is yet to be developed.

3. Rigorous analysis of DSWs and their interactions via the inverse scattering transform (IST). Along with providing rigorous justification of the results from modulation theory, the IST method yields the description of many subtle details not captured by the formal modulation analysis. A number of fundamental results have recently been obtained in the rigorous description of DSW generation and interaction problems for the KdV and defocusing NLS equations. The extension to other integrable systems (modified KdV, vector NLS, etc.) is needed.

4. Incoherent wave structures in dispersive hydrodynamics. This is a new, rapidly developing direction involving the mathematical description of a range of macroscopically incoherent wave structures in dispersive hydrodynamics. The novel lines of investigation include integrable turbulence, soliton gases, incoherent DSWs, effectively viscous fluid dynamics in multidimensional dispersive hydrodynamics systems, and nonlinear wave counterparts of quantum mechanical effects in disordered media (e.g. Anderson localization). A closely related range of dispersive-hydrodynamic problems attracting recent interest concerns the formation of rogue waves in shallow and deep water as well as in nonlinear optical systems.

5. Universality of wave breaking in dispersive hydrodynamics. The rigorous mathematical description of the emergence of a DSW in the space-time vicinity of a gradient catastrophe is of fundamental importance for the foundations of dispersive hydrodynamics. There has been notable progress in proving Dubrovin's Universality conjecture for a number of integrable systems, with many deep, fascinating results obtained in this direction. There are still many unanswered questions, especially related to Universality in non-integrable (including viscously modified) systems.

6. DSWs in focusing media, in nonlocal media and in systems with higher order and nonlinear dispersion. DSW phenomena are not limited to hyperbolic systems modified by weak dispersive terms. The dispersive resolutions of gradient catastrophes can also occur in focusing media as well as in media characterised by nonlocal nonlinearities and higher order dispersive mechanisms. The relevant mathematical models include the focusing NLS equation and its modifications due to non-locality, as well as systems with higher order/nonlinear dispersion arising in nonlinear optics, shallow-water dynamics and other areas. The main mathematical approach in the analysis of the dispersive regularization of gradient catastrophe in the integrable focusing NLS equation is the powerful Riemann-Hilbert steepest descent method, which has enabled a number of recent significant advances in the analytical description of semi-classical "elliptic" dynamics. The outstanding problems in this area include the investigation of higher-order breaking dynamics for analytical data and the evolution of non-analytic data. Of special interest are the recently proposed possible connections between the generation of multiperiodic breather type structures in the resolution of focusing singularities and rogue wave formation. The description of DSWs in nonlocal media (e.g. in liquid crystals) and in systems with higher order/nonlinear dispersion (e.g. fully nonlinear shallow water, nonlinear optics with higher order dispersion) is of great importance for applications and has only recently begun to be explored, posing a number of challenging mathematical questions.

7. Quantitative comparison of results of DSW experiments with existing theories. Despite the numerous existing theoretical results in the area of DSWs, the quantitative experimental verification of existing theory is presently lacking. There are only a handful of experimental results confirming certain features of DSW dynamics but no detailed comparison is available. Such a comparison, however, is vital for the further development of dispersive hydrodynamics as a fundamental discipline that stands on its own much like classical, viscous fluid dynamics.

3 Presentation Highlights

The meeting was organized around a general theme each day. A discussion of the highlights of the presentations on each day follows.

3.1 Day 1: Inverse Scattering Transform and analysis related to DSWs

The presentations on Day 1 mostly encompassed rigorous approaches to the study of dispersive hydrodynamics. These works can be divided into two classes, general methods encompassing non-integrable equations and methods applicable to integrable systems. First, the general methods.

Boris Dubrovin began the meeting with an overview of his general approach to wavebreaking in perturbed Hamiltonian systems. This Universality Conjecture that gradient catastrophe is dispersively resolved for short times by Painlevé transcendants, is an important short-time result for dispersive hydrodynamics. The rigorous proof of this conjecture has been completed for the integrable KdV and NLS equations, while there is numerical evidence of its validity in a wider class of equations [18, 19]. Utilizing Whitham averaging, Sergey Gavrilyuk identifies regions of hyperbolicity in a generalized p-system. He finds it useful to utilize mass-Lagrangian coordinates and derives the Whitham equations in a general form useful for analysis. These are important results for advancing a general approach, e.g., not reliant on integrability, to Whitham theory applied to dispersive hydrodynamics. Dispersive shock waves are difficult to study rigorously. For example, there are no existence results in non-integrable systems. Sylvie Benzoni-Gavage presented her work on a general criterion for the stability of periodic traveling waves in the Euler-Korteweg system, an analog of the general soliton stability criterion due to Grillakis, Shatah, and Strauss [20, 21]. As demonstrated previously, there is a direct connection between the hyperbolicity of the Whitham modulation equations and the spectral stability of periodic traveling waves [22].

The other analytical approaches to dispersive hydrodynamics principally involved integrable systems. In an effort to understand the dynamics of the Benjamin-Ono (BO) equation in the small dispersion limit, Peter Miller presented his work on estimating the direct scattering data for the BO equation in the small dispersion regime [23]. Ted Johnson utilized characteristic coordinates in the zero dispersion Ostrovsky equations in order to obtain precise conditions for wavebreaking via the zero of a Jacobian [24, 25]. There were four talks on Nonlinear Schrödinger (NLS) equations, all related to problems with nonzero boundary conditions at infinity. Utilizing the Riemann-Hilbert steepest descent method, Robert Jenkins considers the problem of an initial step for the defocusing NLS equation. The direct scattering is computed explicitly and the steepest descent method is used to provide error estimates that show that the well-known Gurevich-Pitaevskii self similar solution of the Whitham equations does indeed describe the leading order DSW dynamics. Barbara Prinari described the inverse scattering theory for defocusing, three-component (vector) NLS equations with nonzero boundary conditions at infinity [26]. She constructed dark-bright soliton solutions that exhibit a non-trivial polarization shift in the bright components much like bright soliton solutions of the focusing vector NLS equations. Gino Biondini and Dionyssis Mantzavinos presented their work on the focusing NLS equation with nonzero boundary conditions at infinity. Utilizing inverse scattering transform techniques, they identify the long time asymptotics, i.e., the nonlinear stage of modulational instability or what could be called elliptic (subsonic) dispersive hydrodynamics in contrast to the defocusing NLS equation that exhibits hyperbolic (supersonic) dispersive hydrodynamics.

3.2 Day 2: DSW experiments and physical applications

The presentations on Day 2 were principally organized around experiments and applications in dispersive hydrodynamics.

Stefano Trillo, one of the leading experimentalists in optics and dispersive hydrodynamics, began the day with a number of experimental results including laboratory observations of DSWs in water waves, optical DSWs exhibiting linear resonance [27], and incoherent optics involving a nonlocal NLS equation that exhibits a DSW in its Fourier transform [28]. In the water waves problem, Trillo showed comparisons between experiment and simulations of the KdV (long waves) and Whitham (full water waves dispersion) equations. The features of the small amplitude trailing edge of the DSW are particularly affected by the details of the dispersion with better agreement between experiment and the Whitham equation. The DSW-linear wave resonance originates from an inflection point in the dispersion relation, an important theme that will arise in other talks. In particular, Peter Engels presented experiments involving Bose-Einstein condensates (BECs), another medium that supports dispersive hydrodynamics. He demonstrated how, utilizing spin-orbit coupling, the dispersion relation can be engineered to have a more general form than the standard parabolic dispersion of "standard" BECs [29].

The remaining talks involved applications. There were four talks involving applications with NLS type equations. Two-dimensional dispersive hydrodynamics were investigated numerically by Arnaldo Gammal in the context of supersonic NLS flow past an obstacle, leading to the generation of oblique solitons, vortices, and linear bow waves [30]. Line dark solitons in the 2D defocusing NLS equation are known to exhibit a transverse instability. Boaz Ilan presented numerical computations involving supersonic flow in a confined channel where dark solitons are shown to be stable when sufficiently confined, having implications to confined, multi-dimensional dispersive hydrodynamics. Luca Salasnich presented his work on a generalized NLS model of a unitary Fermi gas that exhibits DSWs and compared these results with experiment [31]. There is some debate as to whether or not the unitary Fermi gas is effectively regularized by dispersion or dissipation [32]. A sequence of two talks by Gennady El and Alex Tovbis considered the small dispersion regime for the focusing NLS equation with a localized, "box" initial condition [33]. They compare the results of Whitham theory, the Riemann-Hilbert approach to inverse scattering theory, and careful numerical simulations, identifying DSWs and the local generation of approximate breathers that resemble structures studied in the context of rogue waves. Some of these elliptic dispersive hydrodynamics have similarities to hyperbolic dispersive hydrodynamics during the initial wavebreaking, DSW generation processes. However, the long time dynamics result in the generation of more and more phases in the modulated, quasi-periodic solutions. This is in contrast to the typical reduction in phases for the long time dynamics of hyperbolic dispersive hydrodynamics, e.g., for KdV and defocusing NLS.

Discussed so far have been approaches to dispersive hydrodynamics involving Riemann-Hilbert/steepest descent and Whitham averaging. Mark Ablowitz described another approach involving matched asymptotic expansions for the long time dynamics of the KdV equation with differing, constant boundary conditions at infinity [34]. He showed that, no matter how many phases are generated during intermediate times, the long time dynamics of this problem result in, at most, a single-phase DSW at large time plus radiation and solitons. This demonstrates the importance of DSWs as coherent solution components, along with dispersive radiation and solitons, of dispersive hydrodynamics. A talk by Guo Deng presented WKB expansions for the

Schrödinger scattering problem in order to "count" the number of solitons in the seminal Zabusky-Kruskal computations [1]. The result was nine solitons. The talks were wrapped up by Naum Gershenzon who spoke on an application of the sine-Gordon equation and modulation theory to macroscopic friction.

The talks on Day 2 demonstrated the ubiquity of dispersive hydrodynamics in physical applications.

3.3 Day 3: connections between dispersive hydrodynamics and conservation laws

Day 3 was a half day of talks chiefly centered upon the connections between hyperbolic conservation laws and dispersive hydrodynamics.

Leading off the day was Philippe LeFloch, who provided an overview of rigorous and numerical methods for conservation laws regularized by dispersion *and* diffusion [35]. Such regularizations lead to microstructure near singularities and can have a significant impact on the uniqueness of solutions to dissipationless and dispersionless conservation laws. The emphasis in this talk was on the diffusion dominated regime, i.e., when the zero diffusion/dispersion limit exists in a strong sense, as opposed to the existence of only a weak limit in the zero dispersion limit case. Conservation laws with non-convex flux exhibit particularly interesting behavior, including shocks that do not satisfy the standard Lax-Oleinik entropy conditions. Rather, LeFloch seeks Riemann problem solutions that satisfy a single entropy inequality as well as a compatible *kinetic function*. This allows for uniqueness of solutions. Solutions in these systems can include double wave structures and undercompressive shocks. Utilizing these ideas, LeFloch describes numerical schemes that satisfy these requirements to a given order in the grid spacing, yet do not need to resolve the microscale generated by the higher order dispersive terms [36].

An alternative method by which to resolve diffusive-dispersive models was described by Michael Shearer [37]. He focused upon the modified KdV-Burgers equation by considering traveling wave solutions. A complete classification of the Riemann problem can be carried out. The interest here is in the relationship between the diffusion dominated traveling wave analysis and the dispersion dominated regime resulting in unsteady DSWs. A complete classification of the Riemann problem for the dissipationless modified KdV equation can also be carried out using Whitham theory. By direct comparison, Shearer identifies kinks as the purely dispersive analog of diffusive-dispersive undercompressive shocks. A new type of DSW is also identified in the mKdV equation, termed a contact DSW (CDSW) due to the propagation of its soliton edge with the local characteristic wave speed. A multi-valued mapping between diffusive-dispersive and purely dispersive Riemann problems showcases the similarities and differences between the two.

Antonio Moro presented a novel approach to understanding phase transitions by interpreting the order parameter of thermodynamic theories as the solution of a dissipatively and/or dispersively regularized conservation law [38, 39]. This yields powerful, direct relationships between shock waves and phase transitions. For example, the triple point between a solid, liquid, and gas can be identified with the merger of two shock waves. Moro goes on to describe a number of specific models that can be analyzed using conservation law methods. He ends his talk with a discussion of complex/disordered systems, e.g., spin glasses, that may be understood utilizing dispersive or diffusive-dispersive regularizations of conservation laws.

The day was completed by Christian Klein with a talk on numerical studies of DSWs in 1D [40] and 2D equations that exhibit blow-up. DSWs can be viewed as the dispersive regularization of singularity formation (gradient catastrophe). A different type of blow-up singularity that is not regularized by dispersion can occur in critical and supercritical dispersive wave equations. The competition between these two mechanisms was described by numerical computations principally of the generalized KdV and generalized KP equations. A key result was the observation that, for the initial data considered, the blow-up time was always greater than the critical time for gradient catastrophe.

3.4 Day 4: soliton turbulence experiments in water waves, optics; dispersive hydrodynamic applications in classical fluids

The final day of the meeting was broken up into two themes. The morning talks emphasized experiments and theory for soliton turbulence. The afternoon talks mainly involved the dispersive hydrodynamics of classical fluids.

Soliton turbulence is the branch of dispersive hydrodynamics that seeks to characterize the statistical properties of a complex, incoherent collection of solitons in a dispersion dominated medium. Alfred Osborne led the morning talks with a presentation on ocean water wave observations that exhibit statistical features consistent with a soliton gas [41]. In particular a $1/f$ dependence of the spectrum on the frequency f is consistent with a KdV soliton gas. In order to carry out this analysis, Osborne has developed numerical tools to analyze time series data using finite gap theory. A novel feature that he has recently observed is that of a gas of focusing NLS breathers. Breathers are commonly identified with rogue wave applications. The next talk by Pierre Suret presented experiments and numerical simulations of random initial waves propagating in a single mode optical fiber accurately modeled by the focusing NLS equation [42]. The experimentally observed, heavy tailed probability density function is shown through numerical simulations to be related to the generation of coherent structures such as breathers. Stéphane Randoux followed with a presentation on intermittency in an optical fiber experiment modeled by the defocusing NLS equation [43]. In contrast to the previous talk, here the probability density function of the incoherent wave amplitude is found to decay faster than typical Gaussian statistics. This phenomenon is identified with intermittency. The final morning speaker presented observations in nonlinear optics of shallow water wave type dynamics. Here, the propagation of a chirped pulse in a normally dispersive fiber (modeled by the defocusing NLS equation) is dominated by the nonlinear, convective terms, i.e., dispersionless terms. The dynamics of the system are well captured by the dispersionless shallow water equations. This result identifies the pre-wavebreaking connection between conservation laws and dispersive hydrodynamics.

The afternoon session focused upon the dispersive hydrodynamics of classical fluids. The first pair of talks by Michelle Maiden and Mark Hoefer involved two Stokes fluids with high viscosity contrast. When a lighter, less viscous fluid is injected from below into a vessel filled with a heavier, more viscous fluid, a stable fluid conduit can form. The interfacial dynamics of this conduit are dominated by nonlinear self-steepening due to buoyancy and nonlinear, nonlocal dispersion due to viscous interfacial stresses. These two features are the key components for dispersive hydrodynamics. Maiden presented a live demonstration and laboratory experiments of solitons, DSWs, and their interactions in this viscous fluid conduit system. The results demonstrate the versatility of this model dispersive hydrodynamic system. Soliton refraction by a DSW and DSW merger/refraction by another DSW are experimentally demonstrated. Hoefer followed Maiden's talk with a theoretical description of the system [44, 45]. Depending on the wavenumber, either the defocusing or focusing NLS equations describe the envelope dynamics of weakly nonlinear, periodic waves. Periodic solutions of the long-wave conduit equation were considered in the context of Whitham theory whereby the modulation equations were shown to be hyperbolic or elliptic depending on the wavenumber and amplitude of the carrier wave. The sign of dispersion is shown to change dynamically in the process of DSW formation whereby a multi-phase, implosion region is generated. This demonstrates the role of dispersion convexity, building on previous talks where the flux convexity played an important role. The next talk by Gavin Esler provided an analysis of the Riemann problem to the Miyata-Choi-Camassa system of equations modeling a Boussinesq two-layer fluid [46]. Using Whitham-El DSW fitting theory, the internal wave dynamics are shown to exhibit solibores (dispersive analogs of undercompressive shocks) and double wave structures. This and the previous talks on viscous fluid conduits, provide examples of the utility of Whitham theory as applied to non-integrable equations. The peculiar features of non-convex flux and/or non-convex dispersion can lead to novel structures.

Karima Khusnutdinova presented work on stratified fluids related to the theme of multidimensional dispersive hydrodynamics [47]. Khusnutdinova derives a cylindrical KdV type equation for the internal wave amplitude of a piecewise constant shear flow. The geometry of these internal ring waves are shown to be different from those on the surface. The model equation can be used to study multidimensional dispersive hydrodynamics in the ocean and other fluids. A fundamental, important aspect of dispersive hydrodynamics is the existence of conservation laws. For example, they can be used to derive Whitham modulation equations. Dimitrios Mitsotakis presented a direct, asymptotic derivation of conservation of mass, momentum, and energy for shallow water flows governed by the Serre equations. Numerical computations of undular bores and shoaling solitary waves exhibit canonical dispersive hydrodynamics.

The final talk of the meeting by Nikola Stoilov was on the construction of an integrable, dispersive regularization of the integrable Witten-Dijkgraaf-Verlinde-Verlinde equations. As shown throughout the meeting, integrable, dispersive nonlinear wave equations allow for powerful analytical methods to describe dispersive hydrodynamics.

4 Scientific Progress Made

Some important themes that emerged during the course of this meeting include

- **Hyperbolicity versus ellipticity:** Strict hyperbolicity or convexity in the dispersionless equations and the Whitham modulation equations yield what may be termed classical dispersive hydrodynamics. Such behavior is well modeled in the KdV and defocusing NLS equations. Talks by Biondini, Mantzavinos, El, Tovbis, Osborne, and Suret showed the emerging problems in soliton turbulence and rogue waves as important examples of dispersive hydrodynamics modeled by elliptic modulation equations. Modulational instability enhances incoherence yet coherent structures play an important role. Additionally, a dynamic change in the type of the modulation equations, analogous to transonic gas dynamics, yield new dispersive hydrodynamic features as shown by Maiden and Hoefer in the viscous fluid conduit system.

- **Non-convex nonlinearity *or* dispersion:** Hyperbolic conservation law theory has been developed since the eighties to resolve non-classical shock structures such as undercompressive shocks. One of this meeting's main themes was to provide a bridge between the conservation law and dispersive nonlinear wave communities. Diffusive-dispersive models described by LeFloch and Shearer demonstrate the similarities and differences between diffusive-dispersive and purely dispersive regularizations when the modulation equations are non-convex or linearly degenerate. The analysis of non-classical DSWs such as kinks or solibores, contact DSWs (also called trigonometric DSWs), and double wave structures benefit both in physical interpretation and mathematically from the rich hyperbolic conservation theory previously developed. With potential applications to statistical mechanics, among others, as shown by Moro, it is clear that further study of these systems is warranted.

  Another form of non-classical dispersive hydrodynamics can emerge when the linear dispersion is non-convex. This theme was captured by the talks of Trillo, Maiden, and Hoefer. Radiating DSWs and DSW multiphase implosion are examples of the dynamics that can result from a dispersion that changes sign. Experiments in ultracold atoms that can now engineer the dispersion relation as shown by Engels motivate further studies of these non-classical phenomena.

- **Soliton turbulence:** This emerging field of incoherent dispersive hydrodynamics is currently being examined in ocean water wave field studies, e.g., Osborne, and nonlinear optical fiber laboratory studies, e.g., Suret, Randoux. The theory of these systems is in its infancy and, motivated by compelling experimental results, will undoubtedly continue to grow.

- **Application areas:** While "traditional" application areas of dispersive hydrodynamics such as nonlinear optics, superfluids, and water waves were well represented at the meeting, broader applications to friction (Gershenzon), viscous fluid conduits (Maiden, Hoefer), spin-orbit coupled BECs (Engels), thermodynamic phase transitions (Moro), and degenerate Fermi gasses (Salasnich) are promising avenues for further development.

- **General mathematical methods:** The mathematical theory of dispersive hydrodynamics has grown from canonical integrable models (KdV and NLS) to non-integrable systems. Talks by Dubrovin, Gavrilyuk, Benzoni-Gavage, Klein, Hoefer, and Esler reveal important general methods. Short time dynamics and the universality of wavebreaking are active areas of research. Whitham theory is not tied to integrable systems so its use more generally paves the way for further descriptions of non-integrable dispersive hydrodynamics. Basic analytical questions of existence and stability are wide open. For example, is hyperbolicity of the Whitham modulation equations sufficient for DSW stability? More mathematical and numerical tools to study dispersive hydrodynamics are needed.

- **Multidimensional dispersive hydrodynamics:** This area of research is wide open as there are very few results. The meeting exhibited talks by Klein, Gammal, Ilan (numerical studies) and Khusnutdinova (derivation of reduced equations).

- **Revisiting classical results:** Even models as fundamental as KdV and NLS continue to receive new analysis and hence new insights into classical dispersive hydrodynamics. One benefit of these and other integrable models is the detailed analysis enabled by Riemann-Hilbert, steepest descent (Jenkins, Mantzavinos, Tovbis), inverse scattering (Miller, Biondini, Prinari, Deng) and matched asymptotics (Ablowitz).

- **Dispersive hydrodynamics experiments:** There were seven experimental talks (Trillo, Engels, Osborne, Suret, Randoux, Wetzel, and Maiden) representing an expansion of physical interest in the field. Further quantitative comparisons between the mathematical modeling and analysis will serve to heighten interest in the field.

5 Outcome of the Meeting

As demonstrated by the diversity of talks during this meeting, dispersive hydrodynamics represents a broad set of physical and mathematical wave problems. The interplay between physics and mathematics was clear throughout the meeting. Applications in classical fluids, optics, ultracold atoms, and other areas motivate further mathematical developments. The strong attendance by almost all participants to each talk plainly demonstrated the broad interests and connections between the participating scientists. An important outcome of this meeting was the initial step to connect the conservation law and dispersive nonlinear waves communities. Another outcome was the bridging of rigorous analysis, applications, and experiments. This meeting may be a springboard to continued growth in the emerging field of dispersive hydrodynamics.

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