Exponential Asymptotics and Water Waves

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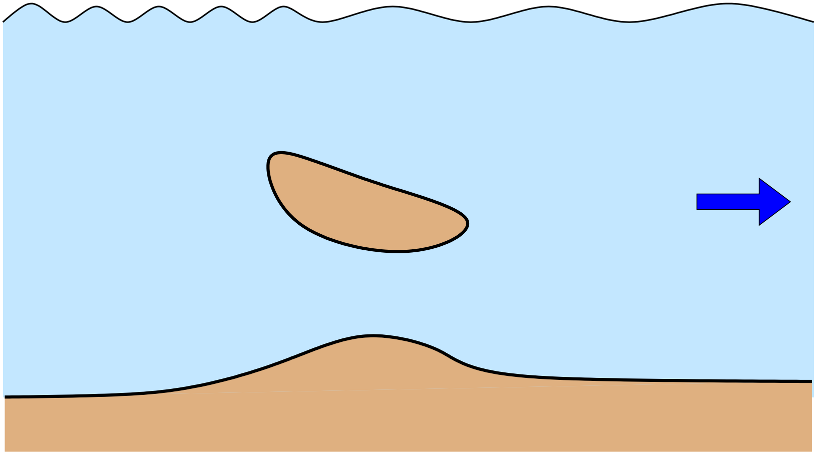
Part I

Gravity-capillary waves in two dimensions

A classical problem in fluid dynamics:
free surface flow over a submerged object/uneven bed.



What is the amplitude of the gravity-capillary waves generated on the free surface?

Consider a steady, two-dimensional, incompressible, irrotational, inviscid flow. The (dimensionless) fluid velocity is $\mathbf{u} = (u, v) = \nabla\phi$, where the velocity potential ϕ satisfies

$$\nabla^2\phi = 0.$$

On all boundaries we have the kinematic condition

$$\frac{\partial\phi}{\partial n} = 0,$$

while on the free boundary we also have the dynamic condition, which from Bernoulli's equation is

$$\frac{1}{2}|\nabla\phi|^2 - \frac{1}{2} + \frac{y}{F^2} = -\delta\kappa$$

where

$$F = \frac{U}{\sqrt{gL}} \text{ (Froude number)}, \quad \delta = \frac{\sigma}{L\rho U^2} \text{ (surface tension parameter)}$$

$L =$ typical lengthscale, $U =$ flow at infinity, $\sigma =$ surface tension

We will take $\delta = 0$.

Such flows have been studied for well over a century (Kelvin 1886; Lamb 1913; Havelock 1927; Gazdar 1973; Dean 1948; Ursell 1950; Wilmott 1987). All these works rely on a linearisation of the free surface conditions, either in an ad hoc fashion, or due to the fact that

$$\left.\begin{array}{l}\text{object diameter} \ll \text{object depth} \\ \\ \text{variation in stream depth} \ll \text{depth}\end{array}\right\} \Rightarrow \text{disturbance to the free stream is small}$$

We will consider the alternative limit $F \to 0$. In that case the free surface is also almost flat, but the disturbance to the free stream is large.

Complex potential $w = \phi + \mathrm{i}\psi$, where $\psi =$ streamfunction. Set $z = x + \mathrm{i}y$. Since w is analytic the map $z \to w$ is a conformal transformation of the flow region to a region of the potential plane. The complex velocity is

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - \mathrm{i}v = q\mathrm{e}^{-\mathrm{i}\theta},$$

where θ is the angle the streamlines make with the x-axis. Then, with s arclength on the free surface,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta,$$

so that differentiating the dynamic condition gives

$$q\frac{\mathrm{d}q}{\mathrm{d}s} + \frac{\sin\theta}{F^2} = 0.$$

Now, since $\psi = 0$ on the free surface,

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\mathrm{d}q}{\mathrm{d}\phi}\frac{\partial\phi}{\partial s} = q\frac{\mathrm{d}q}{\mathrm{d}\phi}.$$

Thus

$$F^2\,q^2\,\frac{\mathrm{d}q}{\mathrm{d}\phi} + \sin\theta = 0.$$

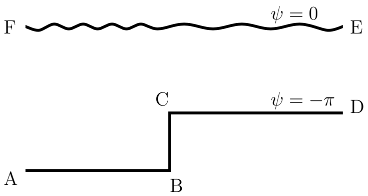
Finally we use another transformation, $w \to \zeta = \xi + \mathrm{i}\eta$ say, to map the potential plane to the upper half ζ-plane. Then, the analyticity of $q\mathrm{e}^{-\mathrm{i}\theta}$ in the upper half ζ-plane implies that on $\eta = 0$,

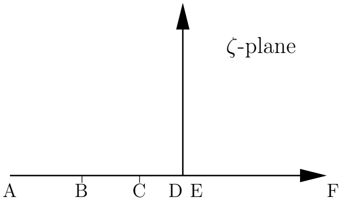
$$\log q = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\theta(\xi')\,\mathrm{d}\xi'}{\xi' - \xi},$$

while the dynamic condition becomes

$$F^2\, q^2 \frac{\mathrm{d}q}{\mathrm{d}\xi} \frac{\mathrm{d}\xi}{\mathrm{d}\phi} + \sin\theta = 0.$$

Example: Flow over a step: $\zeta = \mathrm{e}^{-w}$.





$\psi = -\pi$ corresponds to $\eta = 0$, $\xi < 0$, so that

$$\theta = \begin{cases} 0 & \xi < -a, \\ \pi & -a < \xi < -b, \\ 0 & -b < \xi < 0. \end{cases}$$

Thus

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\xi')\,d\xi'}{\xi' - \xi} = \boxed{\log q = \frac{1}{2}\log\left(\frac{\xi + b}{\xi + a}\right) - \frac{1}{\pi}\int_0^{\infty} \frac{\theta(\xi')\,\mathrm{d}\xi'}{\xi' - \xi}.}$$

On $\eta = 0$,
$$\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\phi} = -\xi\frac{\mathrm{d}}{\mathrm{d}\xi},$$

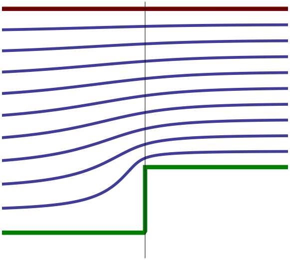
so that the dynamic boundary condition becomes

$$\boxed{F^2\, q^2 \xi \frac{\mathrm{d}q}{\mathrm{d}\xi} = \sin\theta, \qquad \xi > 0.}$$

Denote $F^2 = \epsilon$ and expand

$$\theta = \theta_0 + \epsilon\theta_1 + \cdots, \qquad q = q_0 + \epsilon q_1 + \cdots .$$

then



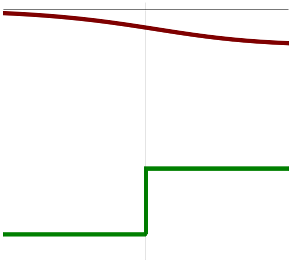
$$\theta_0 = 0, \qquad q_0 = \left(\frac{\xi + b}{\xi + a}\right)^{\frac{1}{2}} .$$

No waves at $O(1)$.

Denote $F^2 = \epsilon$ and expand

$$\theta = \theta_0 + \epsilon\theta_1 + \cdots, \qquad q = q_0 + \epsilon q_1 + \cdots.$$

then



$$\theta_0 = 0, \qquad q_0 = \left(\frac{\xi + b}{\xi + a}\right)^{\frac{1}{2}}.$$

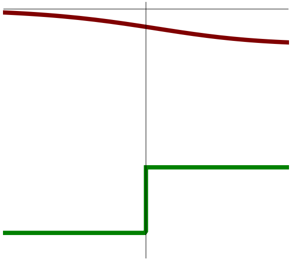
$$q_0^2 \xi \frac{\mathrm{d}q_0}{\mathrm{d}\xi} = \theta_1, \quad q_1 = -\frac{q_0}{\pi} \int_0^\infty \frac{\theta_1(\xi')\,\mathrm{d}\xi'}{\xi' - \xi}.$$

No waves at $O(\epsilon)$.

Denote $F^2 = \epsilon$ and expand

$$\theta = \theta_0 + \epsilon\theta_1 + \cdots, \qquad q = q_0 + \epsilon q_1 + \cdots.$$

then



$$\theta_0 = 0, \qquad q_0 = \left(\frac{\xi + b}{\xi + a}\right)^{\frac{1}{2}}.$$

$$q_0^2 \xi \frac{\mathrm{d}q_0}{\mathrm{d}\xi} = \theta_1, \quad q_1 = -\frac{q_0}{\pi} \int_0^\infty \frac{\theta_1(\xi')\,\mathrm{d}\xi'}{\xi' - \xi}.$$

No waves at $O(\epsilon)$.

In fact, there are no waves at $O(\epsilon^n)$ for any n.

This is known as the low-speed paradox, and was first observed by Ogilvie in 1968.

The waves appear "beyond all orders" of the perturbation expansion in F, and arise as a result of Stokes phenomenon.

The waves are short wavelength and exponentially small in F.

Part II

Asymptotic expansion versus analytic continuation:

Stokes phenomenon

Asymptotic expansion versus analytic continuation

Simplest example is the exponential integral.

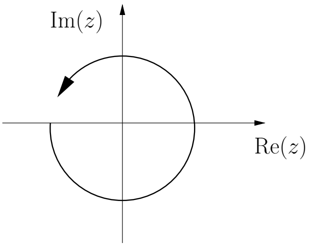
$$\epsilon \frac{\mathrm{d}f}{\mathrm{d}z} + f = \frac{\epsilon}{z}, \qquad f \to 0 \text{ as } z \to -\infty.$$

Using an integrating factor gives

$$f = \mathrm{e}^{-z/\epsilon} \int_{-\infty}^{z/\epsilon} \frac{\mathrm{e}^t}{t} \, \mathrm{d}t \quad \sim \frac{\epsilon}{z} \quad \text{as } \epsilon \to 0 \text{ with } z < 0$$

on integrating by parts.

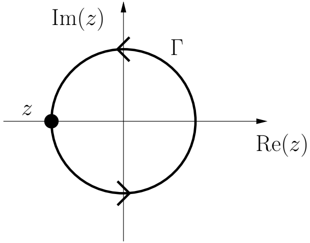
Consider moving z in a circle in the complex plane about zero. Nothing to indicate that $f \sim \epsilon/z$ does not hold everywhere.



However, by the time we get back to $z < 0$ we have

$$f \sim 2\pi \mathrm{i} \mathrm{e}^{-z/\epsilon} + \frac{\epsilon}{z}.$$

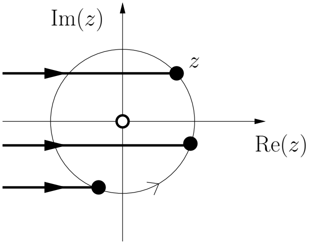
(NB f is not analytic. It has a log singularity at $z = 0$.) How do we know this? From exact solution difference between the values at arg$(z) = -\pi$ and arg$(z) = \pi$ is



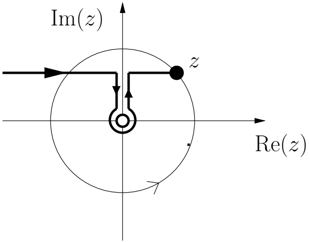
$$\mathrm{e}^{-z/\epsilon} \int_\Gamma \frac{\mathrm{e}^t}{t} \mathrm{d}t = \mathrm{e}^{-z/\epsilon} \times 2\pi \mathrm{i} \times (\text{residue at } t = 0) = 2\pi \mathrm{i} \mathrm{e}^{-z/\epsilon}.$$

Where does this extra term come from? When should we add it?

For complex z we should deform the contour of integration to the steepest descent path:

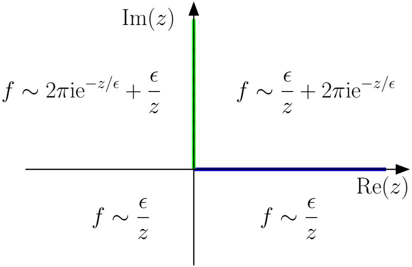


As z crosses the positive real axis this steepest descent path crosses the pole. This is where the extra contribution comes from.



Note that when the new term $2\pi \mathrm{i} e^{-z/\epsilon}$ is turned on across $z > 0$ it is **exponentially small** by comparison to the original ϵ/z.

However, when we cross the imaginary axis to $\mathrm{Re}(z) < 0$ it becomes **exponentially large** and is the dominant term in the expansion of f.



Stokes phenomenon is associated with the steepest descent path crossing a pole, saddle, branch point, etc.

The line in the z plane across which this change happens is called a **Stokes line**.

The line where the dominance switches is called an **Anti-Stokes line**.

What if we don't have an integral representation?

Singular perturbation problems generate **divergent** asymptotic series. Expanding f in powers of ϵ,

$$f \sim \sum_{n=1}^{\infty} \epsilon^n f_n,$$

gives

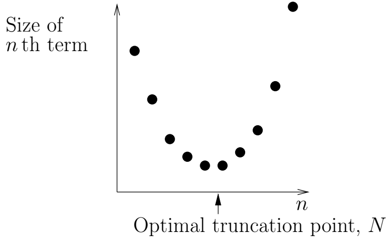
$$\begin{aligned} f_1 &= \frac{1}{z}, \\ \frac{\mathrm{d}f_{n-1}}{\mathrm{d}z} + f_n &= 0, \qquad n \geq 2. \end{aligned}$$

Hence

$$f_2 = \frac{1}{z^2}, \quad f_3 = \frac{2}{z^3}, \quad f_4 = \frac{6}{z^4}, \quad \cdots \quad f_n = \frac{(n-1)!}{z^n}.$$

Zero radius of convergence: Expansion diverges for any fixed ϵ, z.
This is a sign that an exponential is lurking behind the series.

This factorial/power divergence is typical of singular perturbation problems. It will occur when the leading-order solution has a singularity (possibly complex). E.g.



$$f_1 = z^{5/4}, \quad \frac{1}{z^2+1}, \quad \tan^{-1} z, \quad \cdots .$$

Here $f_1 = 1/z$ is singular at $z = 0$.

For a good approximation we need to **truncate** the asymptotic expansion:

$$f \sim \sum_{n=1}^{N-1} \epsilon^n f_n + R_N.$$

For fixed N, $R_N \sim \epsilon^N$ as $\epsilon \to 0$. But if we truncate at the least term (that is, we truncate **optimally**) then R_N is exponentially small.
Ratio between successive terms is $\epsilon n/z$.
Hence least term occurs for $N \sim |z|/\epsilon$.

Optimal truncation (Berry 89)

Let's truncate optimally and study the remainder. We find R_N satisfies

$$\epsilon \frac{\mathrm{d}R_N}{\mathrm{d}z} + R_N = \frac{\epsilon^N (N-1)!}{z^N}. \tag{1}$$

Let $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $N = r/\epsilon + \alpha$ where α bounded as $\epsilon \to 0$. Then using Stirling's formula $n! \sim \sqrt{2\pi}\, n^{n+1/2}\, \mathrm{e}^{-n}$, we find

$$\frac{\epsilon^N (N-1)!}{z^N} \quad \sim \quad \frac{\sqrt{2\pi}\,\epsilon^{1/2}}{r^{1/2}} \frac{e^{-r/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}},$$

as $\epsilon \to 0$. Thus the remainder is exponentially small.
The homogeneous solution of (1) is $R_N = \mathrm{e}^{-z/\epsilon}$. Let us write $R_N = S(z)\mathrm{e}^{-z/\epsilon}$, where S is the **Stokes multiplier**. We also write

$$\frac{\mathrm{d}}{\mathrm{d}z} = -\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}}{r} \frac{\mathrm{d}}{\mathrm{d}\theta}$$

because N is a function of r but not θ.

Then

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} \sim \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\sqrt{2\pi}\, r^{1/2}}{\epsilon^{1/2}} \frac{\mathrm{e}^{-r/\epsilon} e^{r\mathrm{e}^{\mathrm{i}\theta}/\epsilon}}{\mathrm{e}^{\mathrm{i}\theta(r/\epsilon+\alpha)}}.$$

We see $\mathrm{d}S/\mathrm{d}\theta$ is **exponentially small** except at $\theta = 0$, where it is $O(\epsilon^{-1/2})$. Thus there is a **boundary layer** in S near $\theta = 0$. We examine this boundary layer by rescaling $\theta = \delta\bar{\theta}$. Then since

$$\mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}/\epsilon} \sim \mathrm{e}^{r/\epsilon + \mathrm{i}\theta r/\epsilon - \theta^2 r/2\epsilon + \cdots}$$

we find

$$\frac{\mathrm{d}S}{\mathrm{d}\bar{\theta}} \sim \frac{\mathrm{i}\sqrt{2\pi}\,\delta r^{1/2}}{\epsilon^{1/2}} \mathrm{e}^{-\delta^2\bar{\theta}^2 r/2\epsilon}.$$

Hence $\delta = \epsilon^{1/2}$ is the right scaling and

$$\frac{\mathrm{d}S}{\mathrm{d}\bar{\theta}} \sim \mathrm{i}\sqrt{2\pi}\, r^{1/2} \mathrm{e}^{-\bar{\theta}^2 r/2}.$$

The boundary condition we have is $S \to 0$ as $\bar{\theta} \to -\infty$. Hence

$$S = \mathrm{i}\sqrt{2\pi} \int_{-\infty}^{r^{1/2}\bar{\theta}} \mathrm{e}^{-t^2/2}\, dt.$$

Error function smoothing of Stokes discontinuity (Berry 89).

As $\bar{\theta} \to +\infty$ we have

$$S \to \mathrm{i}\sqrt{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\, dt = 2\pi\mathrm{i}.$$

Matching with the "outer" solution away from the Stokes line gives the jump in S across the Stokes line as

$$[S]_{\theta=0-}^{\theta=0+} = 2\pi\mathrm{i}.$$

Thus we have seen explicitly the turning on of $2\pi\mathrm{i}\mathrm{e}^{-z/\epsilon}$ as we cross the Stokes line.

Part III

Exponentially small surface gravity waves

Back to waves...

Limit $\epsilon \to 0$ is singular:

$$\log q = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\theta(\xi')\,\mathrm{d}\xi'}{\xi' - \xi}, \qquad \epsilon q^2 \frac{\mathrm{d}q}{\mathrm{d}\phi} + \sin\theta = 0.$$

Expansion

$$\theta \sim \sum_{n=0}^{\infty} \epsilon^n \theta_n, \qquad q \sim \sum_{n=0}^{\infty} \epsilon^n q_n,$$

is divergent.

First step: prepare for analytic continuation by removing the principal value

$$\log q - \mathrm{i}\theta = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\theta(\zeta')\,\mathrm{d}\zeta'}{\zeta' - \zeta}, \qquad \epsilon q^2 \frac{\mathrm{d}q}{\mathrm{d}w} + \sin\theta = 0.$$

Procedure:

- determine the asymptotic powers series expansion in ϵ;
- truncate it optimally;
- formulate the equation for the remainder;
- observe exponentially small terms being switched on across Stokes lines.

In fact, we do not need the full expansion in powers of ϵ (fortunately). Because the optimal truncation point tends to infinity as $\epsilon \to 0$, we only need to know θ_n for large n.

Late terms

Leading-order problem will have singularities as a function of w (through the conformal map ζ). But all higher-order problems are linear. Thus the singular points of q_n are the same as those of q_0. Singular perturbation $\Rightarrow$ factorial/power divergence. We make the ansatz (c.f. JWKB)

$$\theta_n \sim \frac{\Theta\Gamma(n+\gamma)}{\chi^{n+\gamma}}, \qquad q_n \sim \frac{Q\Gamma(n+\gamma)}{\chi^{n+\gamma}}, \qquad \text{as } n \to \infty.$$

At leading order in n we find (eikonal eqn)

$$\mathrm{i}q_0^3 \frac{\mathrm{d}\chi}{\mathrm{d}w} = 1 \quad \Rightarrow \quad \chi = -\mathrm{i}\int_{w_0}^{w} \frac{\mathrm{d}w}{q_0^3},$$

where w_0 is one of the singularities of q_0. At next order we find

$$Q = \frac{\Lambda}{q_0^2} \exp\left(-3\mathrm{i}\int_{w^*}^{w} \frac{q_1}{q_0^4}\,\mathrm{d}w\right).$$

The constants γ and Λ determined by matching with an inner region near w_0.

Stokes line smoothing.

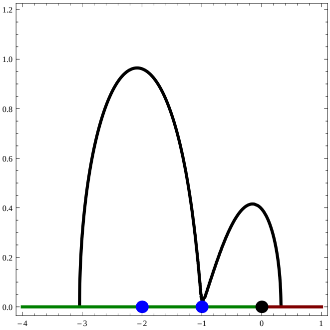
There are Stokes lines whenever χ is real and positive.

Smoothing works in exactly the same way as before, and

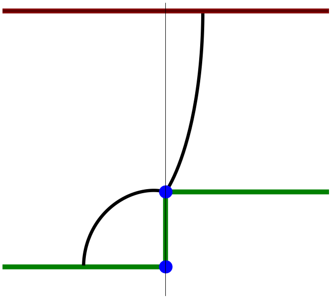
$$\boxed{\theta_{\mathrm{exp}} = \frac{2\pi \mathrm{i}}{\epsilon^{\gamma}} \Theta \mathrm{e}^{-\chi/\epsilon}}$$

is turned on.

Flow over a step



ζ plane



Physical plane

Part IV

Exponentially small surface gravity waves in three dimensions

Example: Free-surface flow over a point source

Denote the free surface by $z = \xi(x, y)$. Then

$$\nabla^2 \phi = 0, \qquad -\infty < z < \xi(x, y),$$

with kinematic and dynamic boundary conditions on $z = \xi$:

$$\xi_x \phi_x + \xi_y \phi_y = \phi_z, \qquad \frac{\epsilon}{2}(|\nabla \phi|^2 - 1) + \xi = 0.$$

Since the flow is uniform in the far field, $\phi_x \to 1$ there, while at the source

$$\phi \sim \frac{\delta}{4\pi\sqrt{x^2 + y^2 + (z+h)^2}} \qquad \text{as} \quad (x, y, z) \to (0, 0, -h).$$

Finally, we require a radiation condition which states that any waves generated must have an outgoing group velocity.

Linearization

We linearize about uniform flow by setting

$$\phi = x + \delta\overline{\phi}, \qquad \xi = \delta\overline{\xi},$$

to give, at leading order in δ, dropping the bars

$$\nabla^2\phi = 0, \qquad -\infty < z < 0,$$

$$\phi_z - \xi_x = 0, \qquad \epsilon\phi_x + \xi = 0, \qquad z = 0,$$

where the boundary conditions are now applied on the fixed surface $z = 0$. The far-field conditions imply that $\phi \to 0$ as $x^2 + y^2 + z^2 \to \infty$, while near the source

$$\phi \sim \frac{1}{4\pi\sqrt{x^2 + y^2 + (z+h)^2}} \qquad \text{as} \quad (x, y, z) \to (0, 0, -h).$$

We analytically continue so that $x, y, z \in \mathbb{C}$, with the free surface still satisfying $z = 0$.

Series expansion

Expanding as a power series in ϵ,

$$\phi \sim \sum_{n=0}^{\infty} \epsilon^n \phi^{(n)}, \qquad \xi \sim \sum_{n=0}^{\infty} \epsilon^n \xi^{(n)},$$

the leading-order solution is simply

$$\phi^{(0)} = \frac{1}{4\pi\sqrt{x^2+y^2+(z+h)^2}} + \frac{1}{4\pi\sqrt{x^2+y^2+(z-h)^2}}, \qquad \xi^{(0)} = 0.$$

Late terms

We again make the factorial-over-power ansatz

$$\phi^{(n)} \sim \frac{\Phi(x,y,z)\Gamma(n+\gamma)}{\chi(x,y,z)^{n+\gamma}}, \quad \xi^{(n)} \sim \frac{\Xi(x,y)\Gamma(n+\gamma)}{\chi(x,y,0)^{n+\gamma}}, \quad \text{as} \quad n \to \infty,$$

where γ is a constant. The singulant, χ, satisfies

$$\chi = 0 \quad \text{on} \quad x^2 + y^2 + (z \pm h)^2 = 0.$$

At leading order as $n \to \infty$, Laplace's equation gives

$$\chi_x^2 + \chi_y^2 + \chi_z^2 = 0,$$

while the boundary conditions on $z = 0$ give,

$$-\chi_z \Phi + \chi_x \Xi = 0, \qquad -\chi_x \Phi + \Xi = 0,$$

there. These have nonzero solutions for Φ, Ξ when $\chi_x^2 = \chi_z$. Thus, on the free surface $z = 0$,

$$\chi_x^4 + \chi_x^2 + \chi_y^2 = 0.$$

This eikonal equation is identical to that obtained by Keller for the case of a surface disturbance. Here though, because the singularity lies below the fluid surface, we must solve for complex x and y with the boundary condition

$$\chi = 0 \quad \text{on} \quad x^2 + y^2 + h^2 = 0.$$

Charpit's method

Writing

$$p = \frac{\partial \chi}{\partial x}, \qquad q = \frac{\partial \chi}{\partial y},$$

Charpit's equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4p^3 + 2p, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2q, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}q}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\chi}{\mathrm{d}t} = 2p^4.$$

Parametrizing the boundary data as

$$x_0 = s,\ y_0 = \pm \mathrm{i}\sqrt{s^2 + h^2},\ \chi_0 = 0, \quad \Rightarrow \quad p_0 = \pm \frac{h}{s},\ q_0 = \pm \frac{\mathrm{i}h\sqrt{s^2 + h^2}}{s^2}.$$

The solution in parametric form is

$$x = s \pm \frac{2h(2h^2 + s^2)t}{s^3}, \quad y = \pm \mathrm{i}\sqrt{s^2 + h^2} \pm \frac{2\mathrm{i}h\sqrt{s^2 + h^2}\, t}{s^2},$$

$$\chi = \frac{2h^4 t}{s^4}.$$

Eliminating t gives

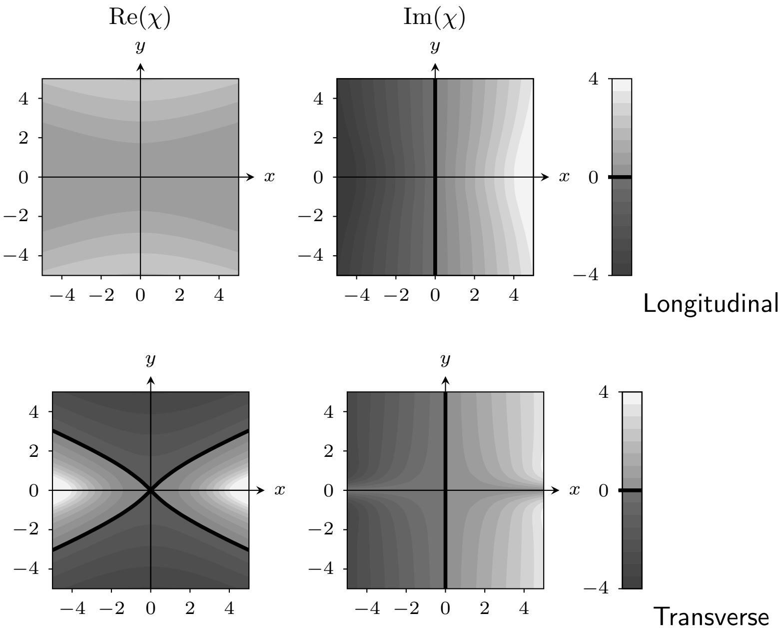
$$\chi = \pm \frac{h^3(s-x)}{s(2h^2+s^2)}$$

where s is any one of the four solutions to

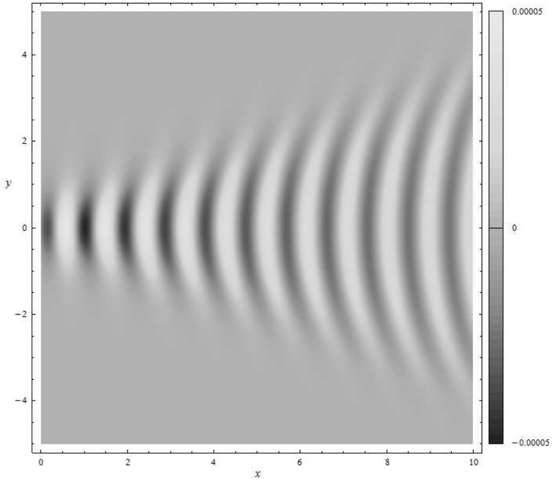
$$\left(x^2+y^2\right)s^4+4xh^2s^3+\left(h^2x^2+4h^2y^2+4h^4\right)s^2+4h^4xs+\left(4y^2h^4+4h^6\right)=0.$$

This gives eight possible singulants. Four may be discounted since they correspond to exponentially large waves at infinity.

The others form two complex conjugate pairs.

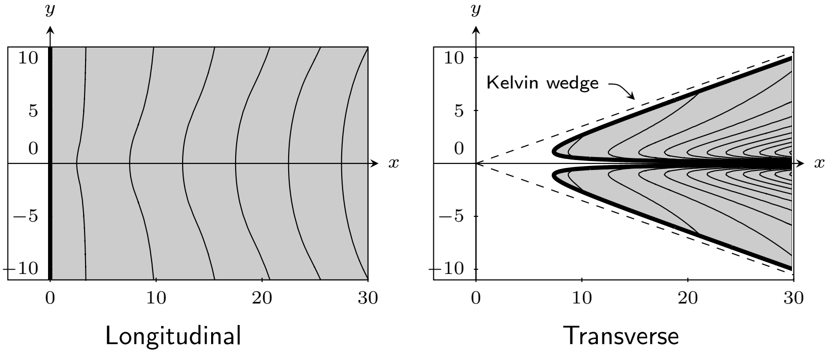


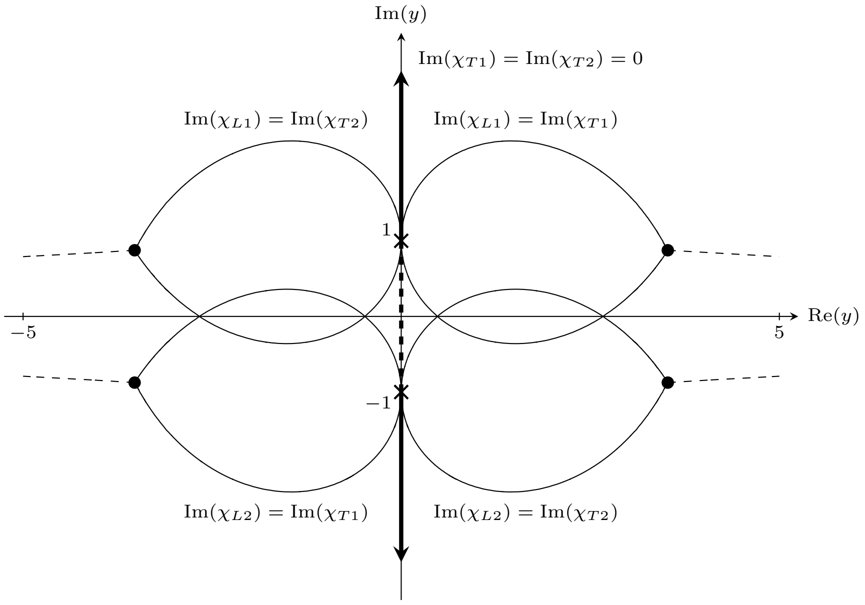
Re(χ)
Im(χ)
y
x
4
2
0
−2
−4
Longitudinal
Transverse



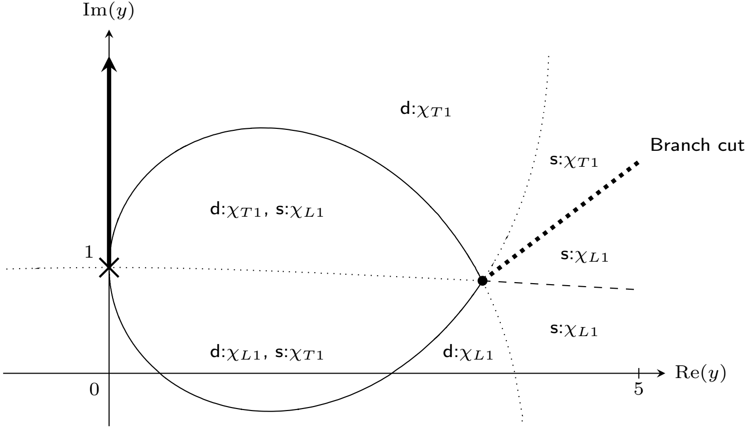
4
2
y
0
−2
−4
0
2
4
6
8
10
x
0.00005
0
−0.00005

Secondary Stokes switching





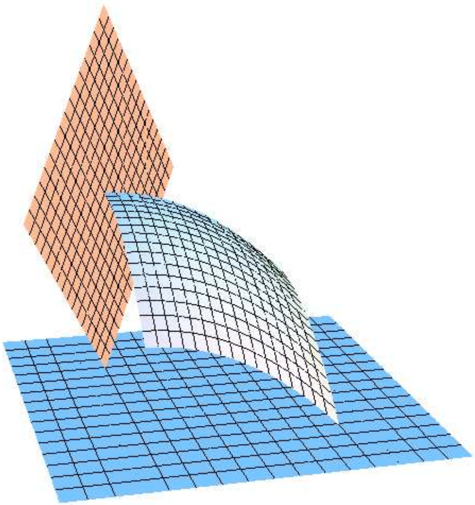
Im(y)
Im(χ_{T1}) = Im(χ_{T2}) = 0
Im(χ_{L1}) = Im(χ_{T2})
Im(χ_{L1}) = Im(χ_{T1})
1
−1
−5
5
Re(y)
Im(χ_{L2}) = Im(χ_{T1})
Im(χ_{L2}) = Im(χ_{T2})



Im(y)
d:χ_{T1}
Branch cut
s:χ_{T1}
d:χ_{T1}, s:χ_{L1}
1
s:χ_{L1}
s:χ_{L1}
d:χ_{L1}, s:χ_{T1}
d:χ_{L1}
Re(y)
0
5

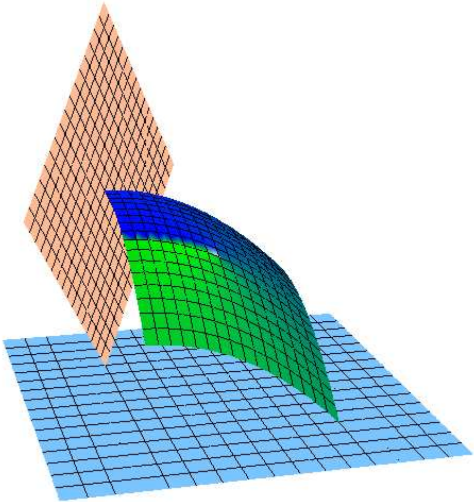
Difficulties

Solving the complex ray equations numerically



Difficulties

Solving the complex ray equations numerically



Summary

- Small Froude number is a singular perturbation
- Singular perturbations generate divergent asymptotic expansions
- Divergent expansions generate exponentially small waves beyond-all-orders via Stokes phenomenon
- Singulant satisfies eikonal equation
- Difficulty in 3D is the topology of $\mathbb{C}^2$ and the numerical solution of the complex ray equations.