The Fokas method for Stokes flow in an L-shaped channel?

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(joint work with Darren G. Crowdy)

Green's Theorem

$$\nabla^2 q = 0 : \qquad \oint_{\partial\Omega} \left[\mathrm{e}^{-\mathrm{i}kz} q_n - q \frac{\partial}{\partial n}(\mathrm{e}^{-\mathrm{i}kz}) \right] ds = 0$$

$$\oint_{\partial\Omega} \mathrm{e}^{-\mathrm{i}kz} \left[q_n + kq \frac{\mathrm{d}z}{\mathrm{d}s} \right] ds = 0,$$

$$\oint_{\partial\Omega} \mathrm{e}^{-\mathrm{i}k\bar{z}} \left[-q_n + kq \frac{\mathrm{d}\bar{z}}{\mathrm{d}s} \right] ds = 0,$$

Dirichlet/Neumann Map (computations by Fornberg & Flyer PRSA 2010, using series of Legendre polynomials modified by appropriate end point factors)

Complex form:

$$\int\int_{\Omega} \left(\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial z} \right) dx dy = \frac{1}{2\mathrm{i}} \oint_{\partial\Omega} (f dz - g d\bar{z})$$

Laplace: $\nabla^2 q = 0$: set $f = \mathrm{e}^{-\mathrm{i}kz} q_z$, $g = 0$

Helmholtz/Modified Helmholtz:

$$\boldsymbol{\nabla}^2 q - 4\lambda q = 0 : \qquad q_{z\bar{z}} - \lambda q = 0 \qquad 4\lambda = \mp\alpha^2$$

$$\left(\exp\left[-\mathrm{i}kz - \frac{\lambda}{\mathrm{i}k}\bar{z}\right] q_z\right)_{\bar{z}} + \frac{\lambda}{\mathrm{i}k}\left(\exp\left[-\mathrm{i}kz - \frac{\lambda}{\mathrm{i}k}\bar{z}\right] q\right)_z = 0$$

Green's Theorem:

$$\oint_{\partial\Omega} \exp\left[-\mathrm{i}kz - \frac{\lambda}{\mathrm{i}k}\bar{z}\right]\left(q_z dz - \frac{\lambda}{\mathrm{i}k} q d\bar{z}\right) = 0$$

Fokas Method for Laplace's Equation in a Polygon

The n-sided polygon has corners at $z_1, z_2, \ldots, z_n$, labeled clockwise

Suggested by complex form of Green's theorem, define

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}kz} q_z dz, \qquad k \in \mathcal{C}, \qquad 1 \le j \le n, \qquad z_{n+1} = z_1$$

Then

$$q_z = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{L_j} \mathrm{e}^{\mathrm{i}kz} \rho_j(k) dk, \qquad L_j = \{k \in \mathcal{C} : \arg k = -\arg(z_j - z_{j+1})\}.$$

$f(z)$ complex-valued, with $f = f_j$ on $[z_j, z_{j+1}]$, $\rho_j(k) = \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}kz} f_j(z) dz$
Then $q(z, \bar{z})$ satisfies Laplace's equation inside polygon and $q_z = f_j$ on $[z_j, z_{j+1}]$.

Verify necessity of L_j definition

$$q_z = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{L_j} \mathrm{e}^{\mathrm{i}kz} \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}k\zeta} q_\zeta d\zeta dk$$

$$(q_z)_{L_j} = \int_{z_{j+1}}^{z_j} \delta(z - \zeta) q_\zeta d\zeta = (q_z)_{L_j}$$

Laplace's equation in a quarter plane

Fokas and Kapeev (2003), Fokas (2008):

$$q_x(x,0)\cos\beta - q_y(x,0)\sin\beta = g_1(x) \qquad 0 < x < \infty$$
$$-q_y(0,y)\cos\beta - q_x(0,y)\sin\beta = g_2(x) \qquad 0 < y < \infty$$
$$q(x,y)\ \text{real}\ , \qquad 0 \le \beta < \pi$$

Set

$$G_1(k) = \tfrac{1}{2}\int_0^\infty \mathrm{e}^{kx} g_1(x)dx, \qquad G_2(k) = \tfrac{1}{2}\int_0^\infty \mathrm{e}^{ky} g_2(y)dy, \qquad \text{Re}(k) \le 0$$

and define the unknown perpendicular derivatives:

$$q_x(x,0)\sin\beta + q_y(x,0)\cos\beta = u_1(x) \qquad 0 < x < \infty$$

$$-q_y(0,y)\sin\beta + q_x(0,y)\cos\beta = u_2(x) \qquad 0 < y < \infty$$

Then

$$\rho_1(k) = \int_0^\infty \mathrm{e}^{-\mathrm{i}kx}\tfrac{1}{2}[q_x(x,0) - \mathrm{i}q_y(x,0)]dx = \mathrm{e}^{\mathrm{i}\beta}[G_1(-\mathrm{i}k) - \mathrm{i}U_1(-\mathrm{i}k)] \qquad \text{Im}(k) \le 0$$

$$\rho_2(k) = -\mathrm{e}^{\mathrm{i}\beta}[G_2(k) - \mathrm{i}U_2(k)] \qquad \text{Re}(k) \le 0$$

where

$$U_1(k) = \tfrac{1}{2}\int_0^\infty \mathrm{e}^{kx} u_1(x)dx, \qquad U_2(k) = \tfrac{1}{2}\int_0^\infty \mathrm{e}^{ky} u_2(y)dy, \qquad \text{Re}(k) \leq 0$$

The Global Relation:

$$\rho_1(k) + \rho_2(k) = 0 \qquad -\pi \leq \arg k \leq -\pi/2$$

is thus

$$G_1(-\mathrm{i}k) - \mathrm{i}U_1(-\mathrm{i}k) = G_2(k) - \mathrm{i}U_2(k) \qquad -\pi \leq \arg k \leq -\pi/2$$

Since q real, apply Schwarz conjugate (conj. followed by $\bar{k} \to k$)

$$G_1(\mathrm{i}k) + \mathrm{i}U_1(\mathrm{i}k) = G_2(k) + \mathrm{i}U_2(k) \qquad \pi/2 \leq \arg k \leq \pi$$

Eliminate U_2

$$2G_2(k) = G_1(-\mathrm{i}k) - \mathrm{i}U_1(-\mathrm{i}k) + G_1(\mathrm{i}k) + \mathrm{i}U_1(\mathrm{i}k) \qquad k \in \mathcal{R}^-$$

$$2G_2(-k) = G_1(\mathrm{i}k) - \mathrm{i}U_1(\mathrm{i}k) + G_1(-\mathrm{i}k) + \mathrm{i}U_1(-\mathrm{i}k) \qquad k \in \mathcal{R}^+$$

Thus

$$\rho_1(k) = \mathrm{e}^{\mathrm{i}\beta}[-2G_2(-k) + 2G_1(-\mathrm{i}k) + G_1(\mathrm{i}k) - \mathrm{i}U_1(\mathrm{i}k)] \qquad k \in \mathcal{R}^+$$

$$\rho_2(k) = \mathrm{e}^{\mathrm{i}\beta}[-2G_2(k) + G_1(\mathrm{i}k) + \mathrm{i}U_1(\mathrm{i}k)] \qquad \pi/2 \leq \arg k \leq \pi$$

Substitute in

$$q_z = \frac{1}{2\pi}\sum_{j=1}^{2}\int_{L_j} \mathrm{e}^{\mathrm{i}kz}\rho_j(k)dk = \frac{1}{2\pi}\left[\int_0^{\infty} \mathrm{e}^{\mathrm{i}kz}\rho_1(k)dk + \int_0^{\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}kz}\rho_2(k)dk\right]$$

Common angle β enables elimination of unknown U_1 by use of Jordan's Lemma

$$\int_{L_1} \mathrm{e}^{\mathrm{i}kz}[G_1(\mathrm{i}k) - \mathrm{i}U_1(\mathrm{i}k)]dk - \int_{L_2} \mathrm{e}^{\mathrm{i}kz}[G_1(\mathrm{i}k) - \mathrm{i}U_1(\mathrm{i}k)]dk = 0$$

Thus

$$q_z = \frac{\mathrm{e}^{\mathrm{i}\beta}}{\pi}\left[\int_0^{\infty} \mathrm{e}^{\mathrm{i}kz}[G_1(-\mathrm{i}k) - G_2(k)]dk + \int_0^{\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}kz}[G_1(\mathrm{i}k) - G_2(k)]dk\right]$$

Interchange of G_1 and U_1 corresponds to $\beta \to \beta \pm \pi/2$ on L_1

Other angles yield a scalar Riemann-Hilbert problem

Biharmonic equation

Fokas and Crowdy (2004)

$$\nabla^4 q = 0 : \qquad q_{zz\bar{z}\bar{z}} = 0$$

$$\frac{\partial}{\partial \bar{z}} \left[q_{zz\bar{z}} + \lambda \left(q_{zz} - \bar{z} q_{zz\bar{z}} \right) \right] = 0 \qquad \text{for all } \lambda$$

Complex form of Green's Theorem:

$$\oint_{\partial\Omega} \mathrm{e}^{-\mathrm{i}kz} q_{zz\bar{z}} dz = 0 \qquad k \in \mathcal{C}$$

$$\oint_{\partial\Omega} \mathrm{e}^{-\mathrm{i}kz} \left(q_{zz} - \bar{z} q_{zz\bar{z}} \right) dz = 0 \qquad k \in \mathcal{C}$$

Define, for $k \in \mathcal{C}$, $1 \leq j \leq n$

$$\rho_{1j}(k) = \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}kz} q_{zz\bar{z}} dz, \qquad \rho_{2j}(k) = \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}kz} \left(q_{zz} - \bar{z} q_{zz\bar{z}} \right) dz$$

Then

$$q_{zz} = \bar{z} Q_1(z) + Q_2(z)$$

where

$$Q_1(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{L_j} \mathrm{e}^{\mathrm{i}kz}\rho_{1j}(k)dk, \qquad Q_2(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{L_j} \mathrm{e}^{\mathrm{i}kz}\rho_{2j}(k)dk,$$

$$L_j = \{k \in \mathcal{C} : \arg k = -\arg(z_j - z_{j+1})\}.$$

$$q_{zz\bar{z}} = Q_1(z) : \qquad \nabla^2 q_{z\bar{z}} = 0$$

$$q_{zz} - \bar{z}q_{zz\bar{z}} = Q_2(z) : \qquad \nabla^2(q_z - \bar{z}q_{z\bar{z}}) = 0$$

Solve as above for $Q_1(z)$ and $Q_2(z)$

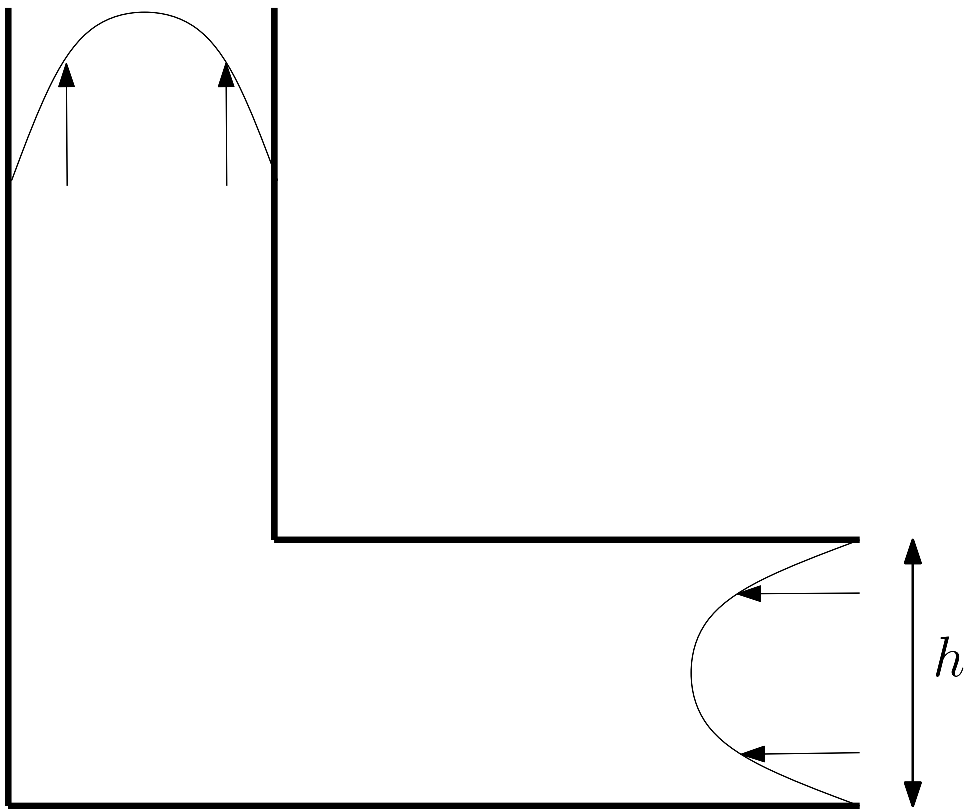
Complex representation

$$q = \text{Im}[\bar{z}f(z) + g(z)] \qquad \nabla^2 q = \text{Im}[4f'(z)]$$

$$q_{zz} = \frac{1}{2\mathrm{i}}\left[\bar{z}f''(z) + g''(z)\right] \qquad q_{zz\bar{z}} = \frac{1}{2\mathrm{i}}f''(z) \qquad q_{zz} - \bar{z}q_{zz\bar{z}} = \frac{1}{2\mathrm{i}}g''(z)$$

Then $q(z, \bar{z})$ satisfies $\nabla^4 q = 0$ inside polygon and $q_{zz} = \frac{1}{2\mathrm{i}}\left[\bar{z}f_j''(z) + g_j''(z)\right]$ on $[z_j, z_{j+1}]$,

The L-shaped channel



$$\boldsymbol{\nabla}^4\psi = 0 \qquad u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

Symmetry about $y = x$

$$v(x,y) + u(y,x) = 0 \qquad \psi(y,x) = \psi(x,y) \qquad p(y,x) = -p(x,y)$$

On line of symmetry

$$v(y,y) = V(y) = -u(y,y) \qquad \boldsymbol{\nabla}^2\psi(y,y) = -2V'(y) \qquad p(y,y) = 0$$

Flow subject to no-slip at channel walls

Incoming pressure-driven flow

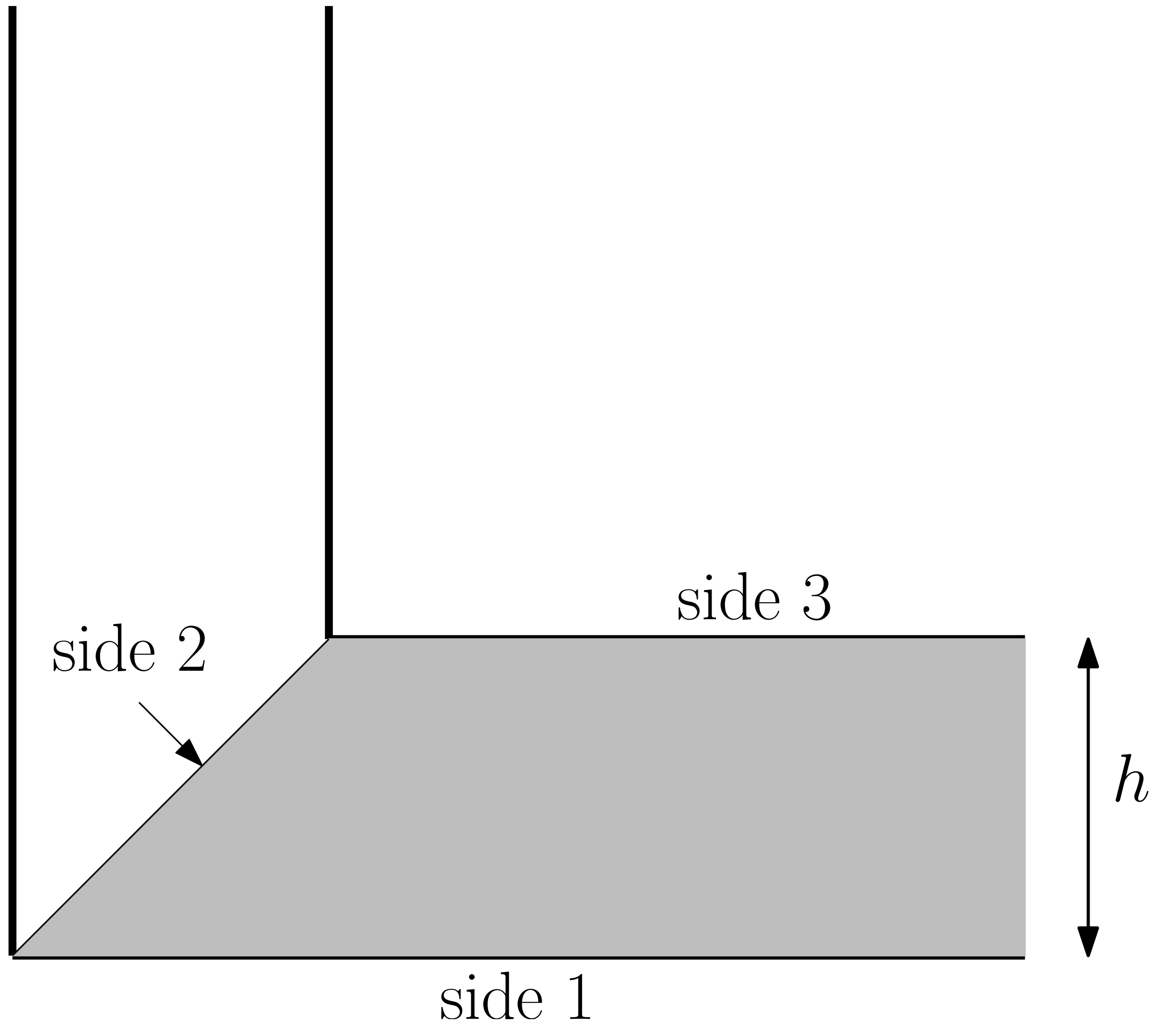
$$u_c = -6y(h-y) \qquad v_c = 0 \qquad p_c = 6\mu(2x-h) + p_0$$

$2p_0$ is additional pressure drop due to corner. Flux $= 2\int_0^h V(y)dy = h^3$

$$u - \mathrm{i}v = -\overline{f(z)} + \bar{z}f'(z) + g'(z)$$

$$f_c(z) = \frac{3}{2}z\left[z - h(1+\mathrm{i})\right] + \frac{p_0 z}{4\mu} + \bar{C} \qquad g_c'(z) = 3z\left(\mathrm{i}h - \tfrac{1}{2}z\right) + C$$

Half-region analysis



Correction to incoming pressure-driven flow

$$f(z) = f_c(z) + f_R(z) \qquad g'(z) = g'_c(z) + g'_R(z) \qquad V(y) = 3y(h-y) + V_R(y)$$

$$f_R(z), g'_R(z) \to 0 \text{ as } x \to \infty \qquad \int_0^h V_R(y)dy = 0$$

No slip at channel walls

$$-\overline{f_R(z)} + zf'_R(z) + g'_R(z) = 0 \qquad z = \bar{z} = x > 0$$

$$-\overline{f_R(z)} + (z - 2\mathrm{i}h)f'_R(z) + g'_R(z) = 0 \qquad z - \mathrm{i}h = x > h$$

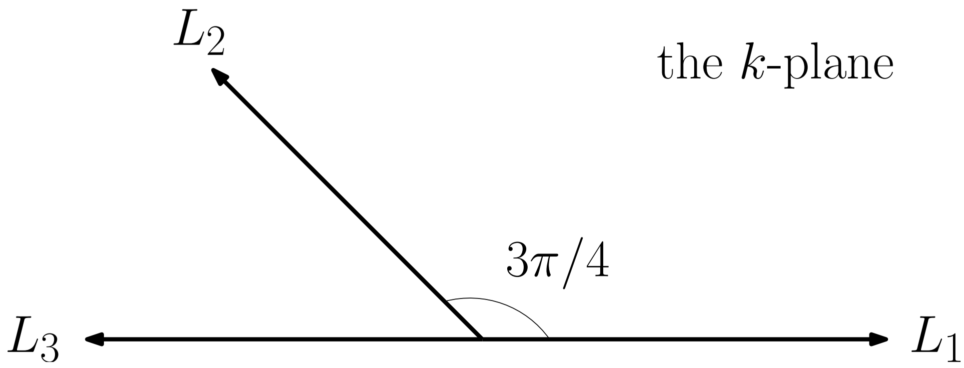
Velocities and stresses at line of symmetry give

$$f_R(y + \mathrm{i}y) = \frac{3}{2}(1+\mathrm{i})y(h-y) - (1+\mathrm{i})\frac{p_0}{4\mu}y + \tfrac{1}{2}(1-\mathrm{i})V_R(y) + \bar{c}$$

$$g'_R(y + \mathrm{i}y) = \frac{3}{2}(1-\mathrm{i})y(2h-y) + \tfrac{1}{2}(1+\mathrm{i})[yV'_R(y) - V_R(y)] + c$$

Evaluate the entire functions

$$\rho_{12}(k) = \int_{(1+\mathrm{i})h}^{0} f_R(z)\mathrm{e}^{-\mathrm{i}kz}dz \qquad \rho_{22}(k) = \int_{(1+\mathrm{i})h}^{0} g'_R(z)\mathrm{e}^{-\mathrm{i}kz}dz$$



$$\begin{bmatrix} f_R(z) \\ g_R'(z) \end{bmatrix} = \frac{1}{2\pi}\left\{\int_{L_1}\begin{bmatrix} \rho_{11}(k) \\ \rho_{21}(k) \end{bmatrix} \mathrm{e}^{\mathrm{i}kz}dk + \int_{L_2}\begin{bmatrix} \rho_{12}(k) \\ \rho_{22}(k) \end{bmatrix} \mathrm{e}^{\mathrm{i}kz}dk + \int_{L_3}\begin{bmatrix} \rho_{13}(k) \\ \rho_{23}(k) \end{bmatrix} \mathrm{e}^{\mathrm{i}kz}dk\right\}$$

$$\begin{bmatrix} \rho_{11}(k) \\ \rho_{21}(k) \end{bmatrix} = \int_0^\infty \begin{bmatrix} f_R(z) \\ g_R'(z) \end{bmatrix} \mathrm{e}^{-\mathrm{i}kz}dz \qquad \begin{bmatrix} \rho_{13}(k) \\ \rho_{23}(k) \end{bmatrix} = \int_{\infty+\mathrm{i}h}^{(1+\mathrm{i})h} \begin{bmatrix} f_R(z) \\ g_R'(z) \end{bmatrix} \mathrm{e}^{-\mathrm{i}kz}dz$$

Global relations

$$\left.\begin{aligned} \rho_{11}(k) + \rho_{12}(k) + \rho_{13}(k) = 0 \\ \rho_{21}(k) + \rho_{22}(k) + \rho_{23}(k) = 0 \end{aligned}\right\} \qquad \mathrm{Im}(k) \leq 0$$

Express no slip conditions in terms of $\rho_{ij}(k)$; eliminate $\rho_{11}, \rho_{21}, \rho_{23}$

$$
\begin{aligned}
(\mathrm{e}^{2kh}-1)\overline{\rho_{13}(-k)} - 2kh\rho_{13}(k) &= \overline{\rho_{12}(-k)} - \rho_{22}(k) + \frac{\mathrm{d}}{\mathrm{d}k}[k\rho_{12}(k)] - \frac{p_0 h^2}{2\mu}\mathrm{e}^{(1-\mathrm{i})kh} \\
&= T(k)
\end{aligned}
$$

Schwarz conjugate

$$
2kh\overline{\rho_{13}(-k)} - (1-\mathrm{e}^{-2kh})\rho_{13}(k) = \rho_{12}(k)\overline{\rho_{22}(-k)} + \frac{\mathrm{d}}{\mathrm{d}k}[k\overline{\rho_{12}(-k)}] - \frac{p_0 h^2}{2\mu}\mathrm{e}^{-(1+\mathrm{i})kh}
$$

Eliminate $\overline{\rho_{13}(-k)}$

$$
\rho_{13}(k) = \frac{khT(k) - \mathrm{e}^{kh}\overline{T(-k)}\sinh kh}{2(\sinh^2 kh - k^2h^2)} = -\frac{T(k)+\mathrm{e}^{kh}\overline{T(-k)}}{4(\sinh kh + kh)} + \frac{T(k)-\mathrm{e}^{kh}\overline{T(-k)}}{4(\sinh kh - kh)}
$$

Numerators must vanish at poles in Im$(k) < 0$; p_0 and $V_R(y)$ to be determined

Strip eigenfunctions evident but Moffatt corner eddies arise from zeros of $\sinh kh + 2kh/\pi$

Fornberg's computations require suitable function expansions on each side

Other subregions

Triangle $0 < y < x < h$

Same conditions on $y = x$ and channel wall section $0 < x < h$

Assume a strip eigenfunction expansion on $x > h$ with unknown coefficients.

Matching of velocities and stresses at $x = h$ is effectively a double condition on the triangle flow

Square $0 < x, y < h$

Exploit symmetry; eliminate diagonal $y = x$ from the analysis

$\psi(y, x) = \psi(x, y)$ implies the correspondencies

$$f(z) \to -\mathrm{i}\overline{f(\mathrm{i}\bar{z})} \qquad f'(z) \to -\overline{f'(\mathrm{i}\bar{z})} \qquad g'(z) \to \mathrm{i}\overline{g'(\mathrm{i}\bar{z})}$$

Only four auxiliary functions

$$\begin{bmatrix} \rho_{11}(k) \\ \rho_{21}(k) \end{bmatrix} = \int_0^h \begin{bmatrix} f_R(z) \\ g_R'(z) \end{bmatrix} \mathrm{e}^{-\mathrm{i}kz} dz \qquad \begin{bmatrix} \rho_{14}(k) \\ \rho_{24}(k) \end{bmatrix} = \int_h^{(1+\mathrm{i})h} \begin{bmatrix} f_R(z) \\ g_R'(z) \end{bmatrix} \mathrm{e}^{-\mathrm{i}kz} dz$$