Green’s function for the Laplace-Beltrami operator on a toroidal surface

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This is joint work with Jonathan Marshall
Green’s functions are fundamental objects and are very important in mathematics (potential theory, complex analysis, numerical analysis etc) and physics (fluid mechanics, electrostatics etc).

The simplest case is that of the Euclidean plane, and for this, Green’s function is of course elementary:

\[ \nabla^2 G = -\delta(z - z_0) \Rightarrow G(z, z_0) = -\frac{1}{2\pi} \log |z - z_0|. \]

Green’s function is also well-known for the case of the unit sphere:

\[ \nabla^2 G = \delta(\theta - \theta_0, \phi - \phi_0) - \frac{1}{4\pi} \Rightarrow \]

\[ G(\theta, \phi; \theta_0, \phi_0) = \frac{1}{4\pi} \log(1 - \cos \theta \cos \theta_0 - \sin \theta \sin \theta_0 \cos(\phi - \phi_0)). \]
However, for more ‘complicated’ surfaces, the theory is mathematically more complicated and far fewer explicit results are known.

Ring torus is natural first choice on which to attempt to extend the existing theory (being the simplest of these ‘complicated’ surfaces).

Herein, we will advocate an approach based on stereographic projection combined with conformal mapping theory, and the judicious use of certain special functions.

We will construct Green’s function for the ring toroidal surface in terms of a single planar complex variable.
Green’s function for the Laplace-Beltrami operator on a spherical surface

For a unit sphere $S$, the Laplace-Beltrami operator $\nabla^2_S$ looks like

$$\nabla^2_S = (1 + \zeta \bar{\zeta})^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} = \frac{\zeta \bar{\zeta}}{F^2_S(\zeta, \bar{\zeta})} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}, \quad F_S := \frac{|\zeta|}{(1 + \zeta \bar{\zeta})}.$$

where $\zeta = \cot(\theta/2) \exp(i\phi)$ is the stereographic projection of the unit sphere to the complex $\zeta$-plane.

Green’s function $G_S(\zeta, \bar{\zeta})$ looks like

$$G_S(\zeta, \bar{\zeta}) = \frac{1}{4\pi} \log \left[ \frac{(\zeta - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)}{(1 - \zeta \bar{\zeta})(1 - \zeta_0 \bar{\zeta}_0)} \right],$$

$$= \frac{1}{2\pi} \log |\omega(\zeta, \zeta_0)| + \hat{G}_S(\zeta, \bar{\zeta}).$$

Here, $\omega(\zeta, \zeta_0) = \zeta - \zeta_0$ is the Schottky-Klein prime function associated with the unit $\zeta$-disc, and $\hat{G}_S$ is an explicitly-known function.

First explicit representation of Green’s function $G_T(\zeta, \bar{\zeta})$ for the Laplace-Beltrami operator $\nabla_T^2$ on a ring toroidal surface $T$, in terms of a single complex variable $\zeta$ in a concentric annulus.

We found:

$$\zeta = Q(\theta) \exp(i\phi),$$

$$\nabla_T^2 \equiv \frac{\zeta\bar{\zeta}}{F_T^2(\zeta, \bar{\zeta})} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}},$$

$$G_T(\zeta, \bar{\zeta}) = \frac{1}{2\pi} \log |\omega(\zeta, \zeta_0)| + \hat{G}_T(\zeta, \bar{\zeta}).$$

Here, $\omega(\zeta, \zeta_0) = P(\zeta/\zeta_0)$ is the Schottky-Klein prime function associated with a concentric annulus in the $\zeta$-plane, and $P$, $F_T$ and $\hat{G}_T$ are explicitly-known functions. Note:

1. similarity in the form of the Laplace-Beltrami operators $\nabla_S^2$ and $\nabla_T^2$;
2. similarity in the structure of Green’s functions $G_S$ and $G_T$;
3. appearance of the Schottky-Klein prime function $\omega(\zeta, \zeta_0)$. 
Motivation: vortical flows on the sphere

Considerable body of work on vortical motion on spherical surfaces (largely motivated by the desire to study phenomena on Earth).


For the future? Extend these studies to the torus (and other compact Riemann surfaces of higher genus).
Motivation: quantized vortices and flows of superfluid films on porous media

Exciting new applications! Behaviour of superfluids and quantized vortices on porous media.

“Understanding the dynamics of quantized vortices bears tremendous importance...in spite of much progress made in the last decade, it should be pointed out that the rigorous mathematical study of a large part of this subject on vortex dynamics remains nearly non-existent" (Du, 2003, Contemp. Math.).
Ring torus $\mathcal{T}_{r, R}$

Let $\mathcal{T}_{r, R}$ denote the ring torus of minor radius $r$ and major radius $R$:

The surface of $\mathcal{T}_{r, R}$ consists of points $(z, y, z) \in \mathbb{R}^3$ where

$$x = (R - r \cos \theta) \cos \phi, \quad y = (R - r \cos \theta) \sin \phi, \quad z = r \sin \theta.$$ 

Pappus’ centroid theorem states that the surface area of a surface of revolution generated by rotating a plane curve about an axis external to the curve and on the same plane is equal to the product of the arc length of the curve and the distance travelled by its geometric centroid.

So, the surface area of ring torus $\mathcal{T}_{r, R}$ is

$$A = (2\pi r)(2\pi R) = 4\pi^2 Rr.$$ 

Curvature of the toroidal surface is non-constant (Gaussian and mean curvatures determined from $\det$ and $\text{tr}$ of the shape operator).
The torus $\mathcal{T}_{r, R}$ is an intrinsically doubly-periodic object.

In terms of our complex variable $\zeta$, Green’s function $G(\zeta, \bar{\zeta})$ we shall construct must be doubly-periodic.

Quick aside: Also the interpretation of a torus as a rectangle (‘flat tori’). Solutions for these ‘flat tori’ problems are commonly constructed in terms of elliptic functions in order to capture their required double-periodicity.
Formulation of problem: the Laplace-Beltrami operator $\nabla^2_{T_{r,R}}$ on the surface of $T_{r,R}$

A straightforward exercise reveals that the Laplace-Beltrami operator on the surface of $T_{r,R}$ is

$$\nabla^2_{T_{r,R}} = \frac{1}{r^2(R - r \cos \theta)} \frac{\partial}{\partial \theta} \left[ (R - r \cos \theta) \frac{\partial}{\partial \theta} \right] + \frac{1}{(R - r \cos \theta)^2} \frac{\partial^2}{\partial \phi^2}.$$

We seek Green’s function $G$ for this operator $\nabla^2_{T_{r,R}}$: this is a real-valued function and satisfies the partial differential equation

$$\nabla^2_{T_{r,R}} G = \delta(\theta, \phi; \theta_0, \phi_0) - \frac{1}{4\pi^2 r R}.$$

Note the constant term on the right-hand-side which is required by the Gauss divergence theorem.

**Goal:** Solve for an explicit representation of Green’s function $G$ by transforming the problem to the complex $\zeta$-plane.

Green’s function $G(\zeta, \bar{\zeta})$ will need to be suitably doubly-periodic in $\zeta$. 

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Central to our construction of Green’s function $G$ is the special stereographic projection taking the surface of the torus $\mathcal{T}_{r,R}$ to a concentric annulus in a complex $\zeta$-plane.

To construct this stereographic projection, consider the composition of two separate mappings.

First, stereographically project $\mathcal{T}_{r,R}$ to a planar rectangle.

Akhiezer (1990) shows that the torus $\mathcal{T}_{r,R}$ can be stereographically projected to a rectangle $\mathcal{R}$ of dimensions $2\pi \times L$ in the $Z$-plane through

$$Z = \phi + i \int_{0}^{\theta} \frac{d\theta'}{\alpha - \cos \theta'}$$

where

$$L = \int_{0}^{2\pi} \frac{d\theta'}{\alpha - \cos \theta'} = \frac{2\pi}{\sqrt{\alpha^2 - 1}}, \quad \text{and} \quad \alpha = \frac{R}{r}.$$
Stereographic projection

It is well-known that the conformal map from this rectangle \( \mathcal{R} \) to a concentric annulus \( D_\zeta = \{ \rho < |\zeta| < 1 \} \) with \( \rho = e^{-L} \) is just

\[
\zeta = \exp (iZ) .
\]

Thus, the stereographic projection of the surface of the torus \( \mathcal{T}_{r,R} \) onto the concentric annulus \( D_\zeta \) in a complex \( \zeta \)-plane is

\[
\zeta = Q(\theta) \exp(i\phi) = \exp \left[ - \int_0^\theta \frac{d\theta'}{\alpha - \cos \theta'} \right] \exp (i\phi) ,
\]

or, upon evaluation of the integral,

\[
\zeta = \exp \left[ - \frac{2}{\sqrt{\alpha^2 - 1}} \tan^{-1} \left[ \sqrt{\frac{\alpha + 1}{\alpha - 1}} \tan(\theta/2) \right] \right] \exp (i\phi) .
\]

This stereographic projection is indeed \textit{conformal} and \textit{one-to-one}.  

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Stereographic projection

\[ \zeta : \mathcal{I}_{r, R} \rightarrow D_\zeta \]

\[ \zeta = \exp \left[ -\frac{2}{\sqrt{\alpha^2 - 1}} \tan^{-1} \left[ \sqrt{\frac{\alpha + 1}{\alpha - 1}} \tan\left(\theta/2\right) \right] \right] \exp(i \phi) \]

Exactly analogous in functional form to the stereographic projection to the plane of the sphere, i.e. both are of the form

\[ \zeta = \mathcal{Q}(\theta) \exp(i \phi). \]

Note:
(1) \( \zeta \mapsto e^{2\pi i} \zeta \) corresponds to a rotation through \( 2\pi \) in the \( \phi \)-direction;
(2) \( \zeta \mapsto \rho \zeta \) corresponds to a rotation through \( 2\pi \) in the \( \theta \)-direction.
Stereographic projection

Using the facts that
\[ \frac{\partial}{\partial \theta} \equiv -\frac{1}{\alpha - \cos \theta} \left[ \zeta \frac{\partial}{\partial \zeta} + \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right], \quad \frac{\partial}{\partial \phi} \equiv i \left[ \zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right], \]
we can express the Laplace-Beltrami operator \( \nabla^2 \) in the following nice form:
\[ \nabla^2 = \frac{4|\zeta|^2}{(R - r \cos \theta)^2} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}. \]

It can be shown after some algebra that
\[ F(\zeta, \bar{\zeta}) := R - r \cos \theta \equiv \frac{2r \eta/A^2}{\eta^2 + 2\alpha \eta + 1} \]
so that \( \nabla^2_{T_{r,R}} \) can be expressed purely in terms of \( \zeta \) and \( \bar{\zeta} \):
\[ \nabla^2 = \frac{4|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}, \]
where we have introduced - for convenience - the variable
\[ \eta = |\zeta|^{1/A}, \quad A = \frac{1}{\sqrt{\alpha^2 - 1}}. \]
\[ \nabla^2 = \frac{4|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \]

Note that the function appearing in the above

\[ h(\zeta, \bar{\zeta}) = \frac{F(\zeta, \bar{\zeta})}{|\zeta|} \]

is the \textit{conformal factor}, which gives

\[ ds^2 = h^2(\zeta, \bar{\zeta}) \left( dq^2 + q^2 d\phi^2 \right) \]

where \( ds \) denotes an element of length on the toroidal surface.
We construct Green’s function $G(\zeta, \bar{\zeta})$ as the solution to the PDE

$$\nabla^2_{Tr,R} G(\zeta, \bar{\zeta}) = \delta(\zeta - \zeta_0) - \frac{1}{4\pi^2 r R}$$

by solving separately for the real-valued function $G_1(\zeta, \bar{\zeta})$ solving

$$\nabla^2_{Tr,R} G_1(\zeta, \bar{\zeta}) \equiv \frac{4|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \frac{\partial^2 G_1}{\partial \zeta \partial \bar{\zeta}} = \delta(\zeta - \zeta_0),$$

and the real-valued function $G_2(\zeta, \bar{\zeta})$ solving

$$\nabla^2_{Tr,R} G_2(\zeta, \bar{\zeta}) \equiv \frac{4|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \frac{\partial^2 G_2}{\partial \zeta \partial \bar{\zeta}} = -\frac{1}{4\pi^2 r R}.$$

Then

$$G(\zeta, \bar{\zeta}) = G_1(\zeta, \bar{\zeta}) + G_2(\zeta, \bar{\zeta})$$

is the Green’s function we seek.

We find an explicit representation for this function $G$ in terms of two special functions which we’ll now introduce.
Special function 1: $P(\zeta; \rho)$

For the concentric annulus $\rho < |\zeta| < 1$, introduce the special transcendental function

$$P(\zeta; \rho) = (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^k \zeta)(1 - \rho^k \zeta^{-1}).$$

This is the Schottky-Klein prime function of the concentric annulus (up to a multiplicative constant).

It satisfies the functional relation:

$$P(\rho \zeta; \rho) = -\zeta^{-1} P(\zeta; \rho).$$

Can define the following subsidiary function:

$$K(\zeta; \rho) = \zeta \frac{P'(\zeta; \rho)}{P(\zeta; \rho)}.$$

This in turn satisfies the functional relation:

$$K(\rho \zeta; \rho) = K(\zeta; \rho) - 1,$$

i.e. it is quasi-periodic as $\zeta \mapsto \rho \zeta$. 
The Schottky-Klein prime function

One form of the Schottky-Klein prime function is presented in Baker’s 1897 monograph in the form of an infinite product:

$$\omega(\zeta, \gamma) = (\zeta - \gamma) \prod_{\theta \in \Theta''} \frac{(\zeta - \theta(\gamma))(\gamma - \theta(\zeta))}{(\zeta - \theta(\zeta))(\gamma - \theta(\gamma))}.$$ 

$\Theta'' \subset \Theta$ is a particular subset of the Schottky group $\Theta$ associated with a bounded $M + 1$ connected planar circular domain $D_\zeta$.


**Key features:** $\omega(\zeta, \gamma)$ is:

(1) defined with respect to $\Theta$;
(2) unique;
(3) analytic everywhere outside the $2M$ Schottky circles of $\Theta$;
(4) $\sim (\zeta - \gamma)$ as $\zeta \to \gamma$;
(5) has a known transformation property as $\zeta \to \theta_j(\zeta)$. 
Concentric annulus as a planar model of a ring torus (Schottky double)


The Schottky-Klein prime function/form is of great importance when constructing meromorphic functions with prescribed zeroes and poles on general Riemann surfaces.
The second special function we will employ is the so-called dilogarithm function $\text{Li}_2(\zeta)$ which is defined by

$$\text{Li}_2(\zeta) = -\int_0^\zeta \frac{\log(1-u)}{u} \, du = -\int_0^1 \frac{\log(1-u\zeta)}{u} \, du.$$ 

Using either of these integral representations, one may expand the logarithm in powers of $\zeta$, obtaining the Taylor series expansion for the dilogarithm:

$$\text{Li}_2(\zeta) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^2},$$

which is defined for all $|\zeta| \leq 1$.

The principal branch of the dilogarithm is defined by the integrals above as a single-valued analytic function in the entire $\zeta$-plane, with the exception of branch points at 1 and $\infty$ (we choose the branch cut to be along the real line from 1 to $\infty$).

The integrals may be used to obtain analytic continuations of the dilogarithm for arguments outside the unit circle.
Special function 2: $\text{Li}_2(\zeta)$

A comprehensive discussion of the various properties of the
dilogarithm function is given in the nice paper by Maximon (2003).

There are several remarkable identities involving the
dilogarithm function, several of which involve the golden ratio, see
e.g. Ramanujan (1919).

Lewin (1991) gives 67 dilogarithm identities ("ladders").

Useful property which can be deduced from the Taylor expansion:

$$\text{Li}_2(\overline{\zeta}) = \overline{\text{Li}_2(\zeta)}, \quad |\zeta| < 1.$$  

which in turn can be used to show that

$$\text{Li}_2(c^{-1}\eta) + \text{Li}_2(c^{-1}\eta^{-1}) = \text{Re}[\text{Li}_2(c^{-1}\eta)],$$

where we recall that $\eta = |\zeta|^{1/A}$, and here $c = -\alpha - 1/A$. 

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Propose:
\[
\frac{\partial G_1}{\partial \zeta} = \frac{K(\zeta/\zeta_0; \rho)}{4\pi \zeta} + \frac{\gamma}{2\zeta}.
\]

Why is this? (1) \(\zeta = \mathcal{Q}(\theta) \exp(i\phi)\) and \(\nabla^2_{r, R}\) are of the same form as their spherical counterparts; this ultimately lead to

\[
G_S(\zeta, \zeta_0) \sim \frac{1}{2\pi} \log |\zeta - \zeta_0| \equiv \frac{1}{2\pi} \log |\omega(\zeta, \zeta_0)|, \quad \zeta \to \zeta_0,
\]

for \(\zeta\) in a unit disc. Thus, we should expect

\[
G(\zeta, \zeta_0) \sim \frac{1}{2\pi} \log |P(\zeta/\zeta_0; \rho)| \Rightarrow \frac{\partial G}{\partial \zeta} \sim \frac{1}{4\pi} \frac{K(\zeta/\zeta_0; \rho)}{\zeta}, \quad \zeta \to \zeta_0.
\]

for \(\zeta\) in a concentric annulus.

(2) The additional term \(\gamma/2\zeta\) comes about because, in the analytic extension of \(G(\zeta, \zeta_0)\), one should expect a term of the form \(\log \zeta\). (\(\log \zeta\) is the only non-constant analytic function in a concentric annulus having constant values on the two boundary circles).
(3) Need to check that
\[ \frac{\partial G_1}{\partial \zeta} = \frac{K(\zeta/\zeta_0; \rho)}{4\pi \zeta} + \frac{\gamma}{2\zeta} \]

satisfies
\[ \nabla^2_{\mathcal{T}_r, \mathcal{R}} G_1 = \delta(\zeta - \zeta_0). \]

Done by verifying that
\[ \int \int_T \nabla^2_{\mathcal{T}_r, \mathcal{R}} G_1(\theta, \phi) dA = \int \int_T \delta(\theta, \phi; \theta_0, \phi_0) dA = 1 \]

by transforming variables \((\theta, \phi) \rightarrow (\zeta, \bar{\zeta})\). Can be shown that
\[ \int \int_T \nabla^2_{\mathcal{T}_r, \mathcal{R}} G_1 dA = 4 \int \int_{D_\zeta} \frac{\partial^2 G_1(\zeta, \bar{\zeta})}{\partial \zeta \partial \bar{\zeta}} d\sigma = -2i \oint_{\partial D_\zeta} \frac{\partial G_1(\zeta, \bar{\zeta})}{\partial \zeta} d\zeta \]

and
\[ \oint_{\partial D_\zeta} \frac{\partial G_1(\zeta, \bar{\zeta})}{\partial \zeta} d\zeta = \int_{\partial D_\zeta} \left[ \frac{K(\zeta/\zeta_0; \rho)}{4\pi \zeta} + \frac{\gamma}{2\zeta} \right] d\zeta = \frac{i}{2} \]

which completes the check. (4) Quasi-periodicity...
Find \( \partial G_2 / \partial \zeta \) by integrating directly:

\[
\nabla^2 G_2(\zeta, \bar{\zeta}) \equiv \frac{4\zeta \bar{\zeta}}{F^2(\zeta, \bar{\zeta})} \frac{\partial^2 G_2}{\partial \zeta \partial \bar{\zeta}} = -\frac{1}{4\pi^2 r R}
\]

\[
\Rightarrow \xi(\zeta, \bar{\zeta}) := \zeta \frac{\partial G_2}{\partial \zeta} = -\frac{1}{16\pi^2 r R} \int^{\bar{\zeta}} \frac{F^2(\zeta, \bar{\zeta}')}{\bar{\zeta}'} d\bar{\zeta}'.
\]

After some algebra, it is found that

\[
\xi(\zeta, \bar{\zeta}) = -\frac{i}{8\pi^2} \left[ \log \left( \frac{\eta - c}{\eta - c^{-2}} \right) + \frac{1}{\alpha A} \left( \frac{c}{\eta - c} + \frac{c^{-1}}{\eta - c^{-1}} \right) \right] + \varsigma_1(\zeta).
\]

where \( \eta = |\zeta|^i / A \) and \( c = -\alpha - 1/A \). Thus, the derivative of Green’s function \( G(\zeta, \bar{\zeta}) \) we seek with respect to \( \zeta \) is

\[
\frac{\partial G}{\partial \zeta} = \frac{K(\zeta / \zeta_0; \rho)}{4\pi \zeta} + \frac{\gamma}{2\zeta} + \frac{\xi(\zeta, \bar{\zeta})}{\zeta}.
\]
Integrating

\[
\frac{\partial G_1}{\partial \zeta} = \frac{K(\zeta/\zeta_0; \rho)}{4\pi \zeta} + \frac{\gamma}{2\zeta}
\]

with respect to \( \zeta \) yields

\[
G_1(\zeta, \bar{\zeta}) = \frac{1}{2\pi} \log |P(\zeta/\zeta_0; \rho)| + \gamma \log |\zeta|.
\]

Here, we have chosen the arbitrary function of \( \bar{\zeta} \) of integration in order that \( G_1(\zeta, \bar{\zeta}) \) is indeed a purely real-valued function.
Next, we need to evaluate the integral

\[ \xi(\zeta, \bar{\zeta}) := \zeta \frac{\partial G_2}{\partial \zeta} \Rightarrow G_2(\zeta, \bar{\zeta}) = \int^{\zeta} \frac{\xi(\zeta', \bar{\zeta})}{\zeta'} d\zeta'. \]

After a lot of careful algebra...

\[ G_2(\zeta, \bar{\zeta}) = \lambda_1 \Re[\text{Li}_2(c^{-1} \eta)] + \lambda_2 \log |\eta - c| + \lambda_3 (\log |\zeta|)^2 + \lambda_4 \log |\zeta| + \lambda_5 \]

where the following are real constants:

\[ \lambda_1 = \frac{A}{2\pi^2}, \quad \lambda_2 = -\frac{1}{2\pi^2 \alpha}, \quad \lambda_3 = -\frac{1}{8\pi^2 A}. \]

Notice the dilogarithm function \( \text{Li}_2(\zeta) \) which played a crucial role in determining this integral explicitly.

Recall \( \text{Li}_2(\zeta) \) is made single-valued by choosing a branch cut along the real axis from 1 to \( \infty \).
\( G(\zeta, \zeta_0) \) needs to be doubly-periodic

It is found that for \( G(\zeta, \zeta_0) \) to be suitably real-valued, and invariant as \( \zeta \mapsto e^{2\pi i} \zeta \) and \( \zeta \mapsto \rho \zeta \), we must take

\[
\lambda_4 = -\frac{1}{4\pi}
\]

while \( \lambda_5 \) is real and arbitrary (could be chosen so that \( G(\zeta, \zeta_0) \) satisfies a reciprocity condition). Further, we need

\[
\gamma = \frac{\log |\zeta_0|}{4\pi^2 A}.
\]

Observation: we may express

\[
\log ||\eta|| - c|^2 = -\log h(\zeta, \overline{\zeta}) - \log |\zeta| + \text{constant},
\]

where recall \( h(\zeta, \overline{\zeta}) = F(\zeta, \overline{\zeta})/|\zeta| \) was the conformal factor. From differential geometry, \(-\nabla^2_{Tr,R} \log h(\zeta, \overline{\zeta})\) equals the Gaussian curvature of the toroidal surface.
Green's function $G(\zeta, \zeta_0)$ is found

Green's function $G(\zeta, \zeta_0)$ for the Laplace-Beltrami operator $\nabla^2_{T_r,R}$ on the surface of $T_{r,R}$ is thus found to be

$$G(\zeta, \bar{\zeta}) = \frac{1}{2\pi} \log \left| P \left( \frac{\zeta}{\zeta_0}; \rho \right) \right| + \frac{A}{2\pi^2} \text{Re} \left[ \text{Li}_2 \left( \frac{|\zeta|^{i/A}}{c} \right) \right]$$

$$- \frac{1}{2\pi^2 \alpha} \log \left| |\zeta|^{i/A} - c \right| - \frac{1}{8\pi^2 A} (\log |\zeta|)^2 + \left( \gamma - \frac{1}{4\pi} \right) \log |\zeta|$$

with

$$\gamma = \frac{\log |\zeta_0|}{4\pi^2 A}, \quad c = -\alpha - \frac{1}{A}, \quad A = \frac{1}{\sqrt{\alpha^2 - 1}}, \quad \alpha = \frac{R}{r}.$$ 

It satisfies:

$$\nabla^2_{T_r,R} G(\zeta, \bar{\zeta}) = \delta(\zeta - \zeta_0) - \frac{1}{4\pi^2 r R}.$$ 

It is invariant with respect to the transformations $\zeta \mapsto e^{2\pi i} \zeta$ and $\zeta \mapsto \rho \zeta$ i.e. it is suitably doubly-periodic (single-valued) everywhere on the torus, as is necessary.
Qualitative check: contours of $G(\zeta, \bar{\zeta})$

Contours of $G(\zeta, \bar{\zeta})$ in the annulus $0.19744 < |\zeta| < 1$ for the torus with $r = 1$ and $R = 4$: 

![Contour plots](image)

**Figure 1.** Contour plots of $\psi(\zeta, \bar{\zeta})$ in $D_{\zeta}$ for the torus $\mathcal{T}_{4,1}$ (for which $\rho = 0.19744$), for three distinct values of the singularity $\zeta = \zeta_0$. (a) $\zeta_0 = 0.275$, (b) $\zeta_0 = 0.6$, (c) $\zeta_0 = 0.875$.

Observe the existence of a critical point on the negative real axis, and two saddle points: one located on the positive real axis and another located on the negative real axis close to the boundary.
Qualitative check: contours of $G(\zeta, \bar{\zeta})$

Our (non-exhaustive) numerical experiments ⇒ changing aspect ratio of the torus, the number of saddle and critical points remain the same but does have an effect on the qualitative appearance of the contours.

Future work: rigorous analysis of the behaviour of Green’s function $G(\zeta, \bar{\zeta})$ for the torus!
Application to vortex dynamics

Consider the flow of an infinitesimally thin layer of inviscid, incompressible fluid on a torus.

Key fact: Green’s function $G$ we have just found is the streamfunction describing such a flow with a single point vortex surrounded by a uniform distribution of vorticity.

Velocity field of the fluid $\mathbf{u}$ on the torus is purely tangential to the surface:

$$\mathbf{u} = (0, u_\theta, u_\phi).$$

Owing to the incompressibility condition $\nabla \cdot \mathbf{u} = 0$, introduce streamfunction $\psi(\theta, \phi)$ such that

$$\mathbf{u} = \nabla \psi(\theta, \phi) \times (1, 0, 0) = \left(0, \frac{1}{R - r \cos \theta} \frac{\partial \psi}{\partial \phi}, -\frac{1}{r} \frac{\partial \psi}{\partial \theta}\right).$$
Application to vortex dynamics

\[ u = \nabla \psi(\theta, \phi) \times (1, 0, 0) = \left( 0, \frac{1}{R - r \cos \theta} \frac{\partial \psi}{\partial \phi}, -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right). \]

Velocity field \( u = (0, u_\theta, u_\phi) \) can be expressed in the form

\[ u_\phi - iu_\theta = \frac{2\zeta}{F(\zeta, \bar{\zeta})} \frac{\partial \psi}{\partial \zeta}, \quad F(\zeta, \bar{\zeta}) = \frac{2r\eta/A^2}{\eta^2 + 2\alpha\eta + 1}. \]

Furthermore, it can be shown that the ‘image’ flow in the \( \zeta \)-plane is given by

\[ U - iV = -\frac{2i|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \frac{\partial \psi}{\partial \zeta}. \]
Consider a point vortex on the surface of the torus ⇒ a δ-function distribution of vorticity.

The Gauss divergence theorem ⇒ a single point vortex can’t exist on the torus unless an additional source of vorticity is present, so that the net circulation on \( \mathcal{T}_r,R \) is zero.

**Resolution:** If the point vortex has circulation \(-1\), then endow the torus with a background ‘sea’ of uniform vorticity

\[
\omega_0 = \frac{1}{4\pi^2 r R}.
\]
Let’s now deduce the streamfunction $\psi$ for this system.

Suppose the point vortex is at $(\theta_0, \phi_0)$ on the surface of the torus. Then $\psi$ satisfies

$$\nabla^2 \psi = \delta(\theta, \phi, \theta_0, \phi_0) - \frac{1}{4\pi^2 rR}.$$ 

Thus identify $\psi$ as Green’s function of the Laplace-Beltrami operator on the torus:

$$\psi \equiv G.$$
Velocity field is doubly-periodic

Furthermore:

\[
\begin{align*}
\nu_\phi - i\nu_\theta &= \frac{2\zeta}{F(\zeta, \bar{\zeta})} \frac{\partial G}{\partial \zeta} = \frac{2\zeta}{F(\zeta, \bar{\zeta})} \left[ \frac{1}{4\pi} K(\zeta/\zeta_0; \rho) + \frac{\gamma}{2} + \xi(\zeta, \bar{\zeta}) \right].
\end{align*}
\]

This is invariant as \( \zeta \mapsto e^{2\pi i} \zeta \) and \( \zeta \mapsto \rho \zeta \), as is necessary.

And:

\[
\begin{align*}
U - iV &= -\frac{2i|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \frac{\partial G}{\partial \zeta} = -\frac{2i|\zeta|^2}{F^2(\zeta, \bar{\zeta})} \left[ \frac{K(\zeta/\zeta_0; \rho)}{4\pi \zeta} + \frac{\gamma}{2\zeta} + \frac{\xi(\zeta, \bar{\zeta})}{\zeta} \right].
\end{align*}
\]

Simple pole of \( K(\zeta/\zeta_0; \rho) \) at \( \zeta = \zeta_0 \) gives rise to a singularity reminiscent of a planar point vortex.

Pre-multiplying factor incorporates the curvature of the toroidal surface, and thus generalises the singularity so that it pertains to a point vortex on a torus.

Not obvious whether a single point vortex is itself advected by the velocity field.
Velocity field is doubly-periodic

\[
u_\phi - i u_\theta = \frac{2\zeta}{F(\zeta, \bar{\zeta})} \frac{\partial G}{\partial \zeta} = \frac{2\zeta}{F(\zeta, \bar{\zeta})} \left[ \frac{1}{4\pi} K(\zeta/\zeta_0; \rho) + \frac{\gamma}{2} + \xi(\zeta, \bar{\zeta}) \right].
\]

General arguments in potential theory ⇒ the derivative \( \partial G/\partial \zeta \) to be the sum of two quasi-periodic functions.

Recall that the two functions \( K(\zeta; \rho) \) and \( \xi(\zeta, \bar{\zeta}) \) are indeed quasi-periodic as \( \zeta \mapsto \rho \zeta \):

\[
\frac{1}{4\pi} K(\rho \zeta/\zeta_0; \rho) = \frac{1}{4\pi} (K(\zeta/\rho_0; \rho) - 1), \quad \xi(\rho \zeta, \rho \bar{\zeta}) = \xi(\zeta, \bar{\zeta}) + \frac{1}{4\pi}.
\]

(easy to show invariance as \( \zeta \mapsto e^{2\pi i \zeta} \)).

Thus, the velocity field above can be made doubly-periodic.
(1) The special geometrical nature of the ting toroidal surface has been fully incorporated into our modelling (via analytic conformal projection).

(2) Our result will be able to be readily applied to solve other problems on a ring toroidal surface for which a type of Laplace-Beltrami equation turns out to be the governing equation.

(3) Shed light on interesting phenomena occurring in quantum mechanics and flows of superfluid film vortices on toroidal and more general holey surfaces.

(4) Evidence for developing the new theory for point vortex dynamics on holey surfaces by appealing to Schottky-Klein prime function / form mathematical framework. Tap into branches of geometric function theory such as exterior calculus and Hodge theory to make progress? (currently under investigation by CCG).
\[ G(\zeta, \bar{\zeta}) = \frac{1}{2\pi} \log \left| P \left( \frac{\zeta}{\zeta_0}; \rho \right) \right| + \frac{A}{2\pi^2} \text{Re} \left[ \text{Li}_2 \left( \frac{|\zeta|^{i/A}}{c} \right) \right] \]

\[ - \frac{1}{2\pi^2 \alpha} \log \left| \zeta^{i/A} - c \right| - \frac{1}{8\pi^2 A} (\log |\zeta|)^2 + \left( \gamma - \frac{1}{4\pi} \right) \log |\zeta| \]

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**Main reference:**