Hele-Shaw flows with kinetic undercooling

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Hele-Shaw cell
viscous fingering instabilities

Design studio called *Nervous System*,
http://n-e-r-v-o-u-s.com/blog/.

Artist *Antony Hall*,
http://www.antonyhall.net.
Flow configurations
bubbles propagating, contracting and expanding

(a) Bubble

Ω(t)

β(t)

∂Ω(t)

(b) Channel

Ω(t)

∂Ω(t)

Evolving fluid region Ω(t) in free-boundary Hele–Shaw flow
(a) a bubble geometry, and (b) a channel geometry.
Formulation

important physics in dynamic condition

- Stokes flow averaged over small gap:

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- Nondimensionalise with \( \phi \) representing negative pressure:
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  \[ v_n = \frac{\partial \phi}{\partial n}, \quad x \in \partial \Omega(t) \]

\( \phi \sim \begin{cases} 
\ln |x|, & |x| \to \infty \quad \text{(bubble geometry)} \\
x, & x \to \infty \quad \text{(channel geometry)}
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\sigma \kappa & \quad \text{(surface tension)} \\
cv_n & \quad \text{(kinetic undercooling)}
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what is it?

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- Large Stefan number (or small specific heat) limit of one-phase Stefan problem:

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on solid-melt interface:

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Introduction

Kinetic undercooling
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  on solid-melt interface:


Kinetic undercooling
stability of a straight interface

- Say $x = f(y, t)$, then
  \[ f = f_0(0) - t + \delta e^{\mu t} \cos ny, \]
  where
  \[ \mu = \frac{\mu(0)nt}{1 + cn}. \]

- All modes unstable for injection case (unlike surface tension case, in which higher modes are stabilised).
- How does kinetic undercooling regularise problem?
Part I: Bubble geometry

(a) Bubble

\[ \Omega(t) \]

\[ \beta(t) \]

\[ \partial \Omega(t) \]
Linear stability
perturbing a circular bubble

- Interface \( r = s(\theta, t) \).
- Small perturbation with \( \delta \ll 1 \):
  \[
  s \sim s_0 + \delta \gamma_n(s_0) \cos n\theta.
  \]
- Stability determined by \( G_n(s_0) = \gamma_n(s_0)/s_0 \); turns out
  \[
  G_n(s_0) = G_n(1) \frac{(s_0 + n)^{n-1}}{s_0(1 + n)^{n-1}}.
  \]

- Note for fixed \( s_0 \):
  \[
  G_n(s_0)/G_n(1) \sim 1/s_0
  \]
as \( n \to \infty \).
- Unstable both directions.
- But well-posed.
Conformal mapping

two ideas that we all know about

1. Since $\phi$ satisfies $\nabla^2 \phi = 0$ in $\mathbb{R}^2$,

$$W(z, t) = \phi(x, y, t) + i\psi(x, y, t)$$

must be analytic function of $z = x + iy$ in flow domain.

2. Riemann’s mapping theorem says there exist mapping $z = f(\zeta, t)$ from disc $|\zeta| \leq 1$ to flow domain $\mathbb{R}^2 \setminus \Omega(t)$.

Here $f$ must be univalent and analytic in disc except at a point (which represents $z = \infty$).

Choose this point to be $\zeta = 0$, so that

$$f(\zeta, t) \sim \frac{a(t)}{\zeta} \quad \text{as} \quad \zeta \to 0,$$

where $a(t)$ is real.
Conformal mapping

map from unit disc

\[ z = f(\zeta, t) \]

\[ \nabla^2 \phi = 0 \]

\[ \phi = c v_n \]

\[ v_n = \frac{\partial \phi}{\partial n} \]

\[ \phi \sim \log |z| \]
Differential geometry of curves with complex variables

- Say $\zeta = |\zeta|e^{i\nu}$, so boundary is $|\zeta| = 1$.

- Recall tangent vector $\hat{t} = \frac{dx}{ds}$, where $s$ is arclength.

  In complex variables, this is
  \[
  \hat{t} = \frac{dz}{ds} = \frac{z\nu}{|z\nu|},
  \]
  where $z = f(\zeta, t)$, $\zeta = e^{i\nu}$ gives
  \[
  z\nu = ie^{i\nu}f(\zeta, e^{i\nu}, t)
  \]
  \[
  = i\zeta f\zeta \quad \text{on} \quad |\zeta| = 1.
  \]

- That is, the unit tangent "vector" is
  \[
  \hat{t} = \frac{i\zeta f\zeta}{|\zeta f\zeta|} \quad \text{on} \quad |\zeta| = 1.
  \]
Differential geometry of curves
kinematic condition

- Normal vector is perpendicular to tangent, so

\[ \hat{n} = -i \hat{t} = \frac{\zeta f_\zeta}{|\zeta f_\zeta|} \quad \text{on} \quad |\zeta| = 1. \]

- Velocity in normal direction

\[ v_n = v \cdot \hat{n} = \mathcal{R} \{ v \hat{n} \} = \mathcal{R} \left\{ \frac{f_t \zeta \bar{f}_\zeta}{|\zeta f_\zeta|} \right\}. \]

- Normal derivative

\[ \frac{\partial \phi}{\partial n} = \nabla \phi \cdot n = \mathcal{R} \left\{ \frac{\partial W}{\partial z} \hat{n} \right\} = \mathcal{R} \left\{ \frac{\zeta w_\zeta}{|\zeta f_\zeta|} \right\}. \]

- So kinematic condition \( v_n = \partial \phi / \partial n \) becomes

\[ \mathcal{R} \left\{ f_t \zeta \bar{f}_\zeta \right\} = \mathcal{R} \left\{ \zeta w_\zeta \right\} \quad \text{on} \quad |\zeta| = 1. \]
Complex variable formulation

Polubarinova-Galin equation

- Solution for complex potential in mapped plane is

\[ w \sim \log \zeta + c \mathcal{V}, \]

where \( \mathcal{V} \) is analytic function in disc and whose real part on \( |\zeta| = 1 \) coincides with normal velocity of the boundary:

\[ \mathcal{R} \{ \mathcal{V} \} = v_n = \mathcal{R} \left\{ \frac{f_t \bar{\zeta} f_{\bar{\zeta}}}{|\zeta f_{\bar{\zeta}}|} \right\}. \]

- Kinematic condition gives Polubarinova-Galin equation

\[ \mathcal{R} \{ f_t \bar{\zeta} f_{\bar{\zeta}} \} = 1 + c \mathcal{R} \{ \zeta \mathcal{V}_\zeta \} \quad \text{on} \quad |\zeta| = 1 \]


- Determine \( f \) that satisfies this equation and initial condition.
Bubble geometry

Simple numerical results
four-fold symmetric bubbles

- Mapping will have series expansion

\[ z(\zeta, t) = \frac{a_{-1}(t)}{\zeta} + \sum_{n=0}^{\infty} a_n(t)\zeta^n. \]

Truncate series and enforce PG equation at equally spaced points on circle \(|\zeta| = 1|.

- Numerical results for contracting and expanding bubbles, showing (a,b) the boundary. Both cases exhibit corner formation.

- Curvature \(\kappa\) in (c,d) curvature as a function of \(\nu = \arg \zeta\) over the first quadrant \(0 < \nu < \pi/2\).
Elliptic initial condition

corner persists

(a) The numerical solution (circles) for a contracting bubble with kinetic undercooling, compared to the exact solution from (solid lines) to the simple geometric rule $v_n = 1$.

(b) The corner angle $\theta$ of the exact solution, starting at $\pi$ (no corner), and ending at zero.
Change in scene
Channel geometry

Change in scene

48° turnaround
Part II: Channel geometry
Saffman-Taylor fingers with surface tension
a well-studied problem

- Experiments in channel geometry demonstrate a single finger emerging, of width $\lambda = 1/2$.
- Time-dependent solutions in channel geometry with surface tension (Dylan Lusmore):
  - Isolating single travelling finger, exact solutions with $\gamma = 0$ have arbitrary $\lambda$.
  - Naive perturbation expansion in powers of $\gamma$ still have arbitrary $\lambda$: McClean & Saffman (1981) *J. Fluid Mech.* 102, 455–469.
Discrete families of Saffman-Taylor fingers
discrete families of solutions

- Discrete set of finger widths for each $\gamma$.
- On upper branches, $\alpha^{1/2}((1 + \alpha)^{1/2} - 1) = \gamma^2(n + 7/4)^2$, where $\alpha = (2\lambda - 1)/(1 - \lambda)^2$
Discrete families of Saffman-Taylor fingers
some with exotic shapes

- Discrete set of finger widths for each $\gamma$.
- For sufficiently large $\gamma$, nonconvex finger shapes (here $\gamma = 2.03$).
Channel geometry with kinetic undercooling
travelling fronts and fingers


- Smaller kinetic undercooling \((c < 1)\) gives single finger.

- Selection of \(\lambda = 1/2\) by kinetic undercooling: Chapman & King (2003).

- But numerical results give continuum of solutions. Why?
Streamers caused by applying strong electric field to weakly ionized gas, leading to ionization reaction via collisions of highly energetic electrons with neutral molecules.

- Thin charge layer and associated ionization front forms finger shape.
- Periodic array evolve to travelling finger profiles:
  
Saffman-Taylor fingers with kinetic undercooling

mathematical formulation

- Frame of reference so velocity vanish at nose, complex potential
  \( w(z) = \phi(x, y) + i\psi(x, y) \), where \( \psi \) is streamfunction.

- Conformal map \( \zeta = \xi + i\eta = e^{-\pi w} \) from strip to upper half \( \chi \)-plane. Interface is \( \xi \in [0, 1] \), where \( \xi = 0 \) is tail and \( \xi = 1 \) is nose.

- Redefine kinetic undercooling \( \epsilon = c\pi/2(1 - \lambda) \).

- Dynamic condition and Hilbert transform:

\[
q = 2\epsilon q\xi \cos \theta \frac{d\theta}{d\xi} + \cos \theta,
\]
\[
\log q = -\frac{\xi}{\pi} \int_0^1 \frac{\theta(\xi')}{\xi'(\xi' - \xi)} \, d\xi',
\]

\[
\theta(0) = 0, \quad q(0) = 1, \quad \theta(1) = -\frac{\pi}{2}, \quad q(1) = 0.
\]

- Exact solution for \( \epsilon = 0 \) is

\[
q_0 = \left( \frac{1 - \xi}{1 + \alpha \xi} \right)^{1/2}, \quad \theta_0 = \cos^{-1} q, \quad \alpha = \frac{2\lambda - 1}{(1 - \lambda)^2}.
\]
Saffman-Taylor fingers with kinetic undercooling
the selection problem

- Analytically continue governing equations into $\zeta$-plane.
- Look for asymptotic solution:
  \[ \theta \sim \sum_{n=0}^{\infty} \epsilon^n \theta_n, \quad q \sim \sum_{n=0}^{\infty} \epsilon^n q^n \quad \text{as} \quad \epsilon \to 0. \]
- Can solve explicitly for $\theta_0, q_0, \theta_1, q_1$, but $\lambda$ arbitrary.
- Factorial/power divergence, truncate optimally by considering limit $n \to \infty$.
- Exponentially small remainder switched on across Stokes line that emerges from singularity and intersects real axis.
- Two contributions must exactly cancel, giving solvability condition
  \[ \alpha^{1/2}((1 + \alpha)^{1/2} - 1) = \epsilon(2n + 7/5), \]
  where $n = 0, 1, 2, \ldots$ (discrete branches for fixed $\epsilon$): Chapman & King (2003).
Saffman-Taylor fingers with kinetic undercooling
continuum of solutions

Saffman-Taylor fingers with kinetic undercooling and surface tension too

- Numerical scheme of Mclean & Saffman, Vanden-Broeck, cannot distinguish between analytic fingers and corner-free but not analytic.
- Take limit $\gamma \to 0$.

Here $\epsilon = 0.1$, $\gamma = 0.03$, 0.5, 1.
- Extrapolate to estimate analytic finger with $\gamma = 0$. 
Saffman-Taylor fingers with kinetic undercooling
repeat exercise

- For example, here is \( \epsilon = 0, 0.1, 0.2 \).

- Putting it together:
Saffman-Taylor fingers with kinetic undercooling

Here $\gamma = 0.03$, $\epsilon = 0, 1, 10, 3500$.

Dashed line is semi-circle with unit radius: Chapman & King (2003).
Summary of ideas
if time permits

Hele-Shaw flows amenable to complex variable techniques.
Summary of ideas
if time permits

- Hele-Shaw flows amenable to complex variable techniques.
- Bubbles with kinetic undercooling unstable (but well-posed) in both directions, with corners developing and persisting.
- Exponentially small terms in hyperasymptotic expansion responsible for selecting discrete families of fingers, for both surface tension and kinetic undercooling.
- Numerical solution of fingers with kinetic undercooling bit subtle: we add surface tension and take away.
• Hele-Shaw flows amenable to complex variable techniques.

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• Numerical solution of fingers with kinetic undercooling bit subtle: we add surface tension and take away.

• Exotic shapes of Saffman-Taylor fingers with surface tension on higher branches.
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See:


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