Singular integro-differential equations for a new model of fracture with a curvature-dependent surface tension

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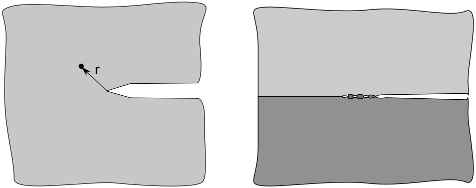
Motivation

Linear elastic fracture mechanics (LEFM)



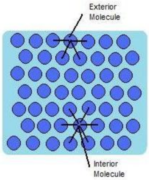
Stress singularities at the crack tips:

- $\sigma + i\tau \sim Cr^{-1/2}$ for a non-interface crack;
- $\sigma + i\tau \sim Cr^{-1/2+i\gamma}$ for an interface crack.

Alternative approaches

- Cohesive and processing zones (Barenblatt, 1959; Dugdale, 1960; Hillerborg 1976).
- Atomistic and lattice based approaches (Marder and Gross, 1995; Abraham et al, 1997; Holland and Marder, 1998).
- Atom-to-continuum approaches (Tadmor et al, 1996; Xiao and Belytschko, 2004).
- Peridynamics (Silling 2000).
- Surface elasticity/surface tension (Gurtin and Murdoch, 1975; Steigmann and Ogden, 1997, 1999; Slattery et al, 2007; Schiavone at al, 2009-2011; Sendova and Walton, 2010; Zemlyanova and Walton, 2012; Zemlyanova, 2013).

Curvature-dependent surface tension model



The linear elasticity model is assumed for the behavior of material in the bulk. A curvature-dependent surface tension acts on the boundary of the crack.

First introduced: E.-S. Oh, J. R. Walton, J.C. Slattery, 2006.
T. Sendova, J. Walton, 2010 for a straight crack.

Advantages:

- Based on physically valid assumptions
- No ad hoc choices of parameters or material properties
- Linear elasticity techniques are valid
- Can be incorporated into industrial FEM codes

New results

The curvature-dependent surface tension model has been applied to:

- Straight non-interface crack (Sendova and Walton, 2010).
- Curvilinear smooth cracks of arbitrary shape (Zemlyanova and Walton, 2012).
- Straight interface crack (Sendova and Walton, 2010; Zemlyanova, 2013).
- Curvilinear interface cracks (submitted).
- Contact problems for a rigid stamp (forthcoming).

Main conclusion: No power singularities of the order $1/2$ or oscillating singularities. Some components of stresses and strains may have logarithmic singularities.

Boundary condition

Surface of the crack is subjected to the surface stress:

$$\mathbf{T}^{(\zeta)} = \tilde{\gamma}\mathbf{P},$$

where $\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is a projection tensor.

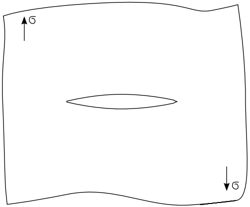
Jump momentum balance condition:

$$\text{grad}_{(\zeta)}\tilde{\gamma} + 2\tilde{\gamma}H\mathbf{n} + [[\ \mathbf{T}\ ]]\mathbf{n} = 0,$$

where

- $\mathbf{n}$ is the unit normal pointing in the bulk of material;
- $\text{div}_{(\zeta)}$ and $\text{grad}_{(\zeta)}$ are the surface divergence and gradient;
- $H = -\frac{1}{2}\text{div}_{(\zeta)}\mathbf{n}$ is the mean curvature;
- $\tilde{\gamma}$ is the surface tension;
- $\mathbf{T}$ is the Cauchy stress tensor.

A straight non-interface crack (Sendova&Walton, 2010)



Linear dependence of the surface tension on the mean curvature:

$$\tilde{\gamma}(x) = \gamma_0 + \gamma_1 \text{div}_{(\zeta)} \mathbf{n}.$$

Linearizing the boundary condition under assumption that $u_{i,j}$, $u_{i,jk}$ are small:

$$\sigma_{12}(x,0) = \gamma_1 u_{2,111}(x,0), \quad \sigma_{22}(x,0) = -\gamma_0 u_{2,11}(x,0).$$

Mathematical techniques

Dirichlet-to-Neumann mappings:

$$\sigma_{12}(x,0) = \alpha_1 u_{2,1}(x,0) + \beta_1 \int_{-\infty}^{\infty} \frac{u_{1,1}(r,0)dr}{r-x};$$

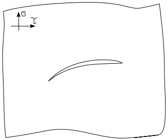
$$\sigma_{22}(x,0) = -\alpha_1 u_{1,1}(x,0) + \beta_1 \int_{-\infty}^{\infty} \frac{u_{2,1}(r,0)dr}{r-x}.$$

Substituting into the boundary conditions leads to the integro-diffrential equation:

$$\gamma_0 u_{2,11}(x,0) + \frac{1}{\pi} \int_{-l}^{l} \frac{\delta_1 u_{2,1}(r,0) + \delta_2 \gamma_1 u_{2,111}(r,0)}{r-x} = -\sigma, \quad x \in (-l,l).$$

Main conclusion: For a curvature-dependent surface tension ($\gamma_1 \neq 0$) stresses and strains remain bounded at the crack tips.

A non-interface crack of arbitrary shape



Linear dependence of surface tension on the change of curvature

$$\tilde{\gamma} = \gamma_1(\mathrm{div}_{(\zeta)}\mathbf{n} - \mathrm{div}_{(\zeta_0)}\mathbf{n}_0).$$

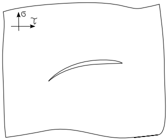
Linearized boundary condition:

$$(\sigma_n + i\tau_n)^{\pm}(s) = \frac{\gamma_1}{2\mu}\kappa_0(s)\left\{\frac{d}{ds}\,\mathrm{Im}(u_1 + iu_2)_t^{\prime\pm} - \kappa_0(s)\,\mathrm{Re}(u_1 + iu_2)_t^{\prime\pm}\right\}$$

$$+i\frac{\gamma_1}{2\mu}\frac{d}{ds}\left\{\frac{d}{ds}\,\mathrm{Im}(u_1 + iu_2)_t^{\prime\pm} - \kappa_0(s)\,\mathrm{Re}(u_1 + iu_2)_t^{\prime\pm}\right\}$$

where "$\pm$" denotes an upper or a lower side of the crack, s is an arc length, $\kappa_0(s)$ is the initial curvature of the crack.

Muskhelishvili's formulas



Stresses and strains in a cracked plane can be expressed through two analytic functions $\Phi(z)$ and $\Psi(z)$ (Muskhelishvili complex potentials):

$$(\sigma_n + i\tau_n)(t) = \Phi(t) + \overline{\Phi(t)} + \frac{d\bar{t}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)}),$$

$$2\mu\frac{d}{dt}(u_1 + iu_2) = \kappa\Phi(t) - \overline{\Phi(t)} - \frac{d\bar{t}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)}),$$

where μ and κ are elastic constants of the cracked plate.

Savruk's Integral Representations

Complex potentials $\Phi(z)$ and $\Psi(z)$ can be written using Savruk's integral representations:

$$\Phi(z) = \Gamma + \frac{1}{2\pi}\int_L \frac{g'(t)dt}{t-z} + \frac{(\kappa+1)^{-1}}{\pi i}\int_L \frac{q(t)dt}{t-z},$$

$$\Psi(z) = \Gamma' + \frac{1}{2\pi}\int_L \left(\frac{\overline{g'(t)dt}}{t-z} - \frac{\bar{t}g'(t)dt}{(t-z)^2}\right)$$

$$+\frac{(\kappa+1)^{-1}}{\pi i}\int_L \left(\frac{\kappa\overline{q(t)dt}}{t-z} - \frac{\bar{t}q(t)dt}{(t-z)^2}\right),$$

where

$$g'(t(s)) = \frac{2\mu}{i(\kappa+1)}\frac{d}{dt}\left((u+iv)^+(s) - (u+iv)^-(s)\right),$$

$$2q(t(s)) = (\sigma_n + i\tau_n)^+(s) - (\sigma_n + i\tau_n)^-(s),\ \ s \in [0, l],$$

are the jumps of stresses and derivatives of displacements.

Boundary conditions:

$$(\sigma_n + i\tau_n)^{\pm}(s_0) = -\frac{\gamma_1}{2\mu}\kappa_0(s_0)\left[\kappa_0(s_0)\,\mathrm{Re}\,\Omega^{\pm}(s_0) - \mathrm{Im}\,\frac{d}{ds_0}\Omega^{\pm}(s_0)\right]$$

$$-i\frac{\gamma_1}{2\mu}\frac{d}{ds_0}\left[\kappa_0(s_0)\,\mathrm{Re}\,\Omega^{\pm}(s_0) - \mathrm{Im}\,\frac{d}{ds_0}\Omega^{\pm}(s_0)\right] + f(s_0),$$

where

$$(\sigma_n + i\tau_n)^{\pm}(s_0) = \pm q(s) + \frac{(\kappa+1)^{-1}}{\pi}\int_0^l \frac{g'(s)ds}{s-s_0} - \frac{(\kappa-1)}{\pi i(\kappa+1)}\int_0^l \frac{q(s)ds}{s-s_0}$$

$$+(\text{regular terms}),$$

$$\Omega^{\pm}(s_0) = \pm\frac{i(\kappa+1)}{2}g'(s_0) + \frac{\kappa-1}{2\pi(\kappa+1)}\int_0^l \frac{g'(s)ds}{s-s_0} + \frac{2\kappa}{\pi i(\kappa+1)}\int_0^l \frac{q(s)ds}{s-s_0}$$

$$+(\text{regular terms}).$$

System of Cauchy singular integro-differential equations

The problem is reduced to the system of two equations:

$$\frac{1}{\pi}\int_0^l \frac{\operatorname{Im} g'(s)ds}{s-s_0} + \frac{\kappa-1}{\pi(\kappa+1)}\int_0^l \frac{\operatorname{Re} q(s)ds}{s-s_0} +$$

$$\frac{\gamma_1}{4\mu}\frac{d}{ds_0}\left\{\kappa_0(s_0)\left[\frac{\kappa-1}{\pi}\int_0^l \frac{\operatorname{Re} g'(s)ds}{s-s_0} + \frac{4\kappa}{\pi(\kappa+1)}\int_0^l \frac{\operatorname{Im} q(s)ds}{s-s_0}\right] +\right.$$

$$\left.\frac{d}{ds_0}\left[\frac{4\kappa}{\pi(\kappa+1)}\int_0^l \frac{\operatorname{Re} q(s)ds}{s-s_0} - \frac{\kappa-1}{\pi}\int_0^l \frac{\operatorname{Im} g'(s)ds}{s-s_0}\right]\right\}$$

$$= \operatorname{Im} M_1(g', q)(s_0), \quad s_0 \in [0, l],$$

where $M_1(g', q)(s_0)$ is a regular linear integral operator.

Unknown functions

4 real unknown functions:

$$\mathrm{Re}\, g'(s),\ \ \mathrm{Im}\, g'(s),\ \ \mathrm{Re}\, q(s),\ \ \mathrm{Im}\, q(s).$$

2 additional real conditions:

$$q(s_0) + \overline{q(s_0)} =$$

$$-\frac{i\gamma_1}{4\mu}\kappa_0(s_0)\left[\kappa_0(s_0)(g'(s_0) - \overline{g'(s_0)}) + i(g''(s_0) + \overline{g''(s_0)})\right];$$

$$q(s_0) - \overline{q(s_0)} = \frac{\gamma_1}{4\mu}\frac{d}{ds_0}\left[\kappa_0(s_0)(g'(s_0) - \overline{g'(s_0)}) + i(g''(s_0) + \overline{g''(s_0)})\right]$$

$$= i\frac{d}{ds_0}\left[\frac{1}{\kappa_0(s_0)}(q(s_0) + \overline{q_0(s_0)})\right].$$

Two words about regularization

Step 1: Both equations of the system have the form

$$\frac{1}{\pi i}\int_0^l \frac{\varphi(s)ds}{s-s_0} = g(s_0), \quad s_0 \in [0,l].$$

Invert the singular integral, obtain weakly singular integro-differential equations.

Step 2: Recast the highest order derivatives as the new unknown functions:

$$\varphi(s_0) = \mathrm{Im}\, g''(s_0); \quad \psi(s_0) = \mathrm{Re}\, q''(s_0).$$

Obtain two weakly-singular integral equations for the new unknowns.

Boundedness of the stresses and the derivatives of the displacements

Rewrite the boundary condition in the form:

$$(\sigma_n + i\tau_n)(s_0) = \frac{\gamma_1}{2\mu}\kappa_0(s_0)\left\{\frac{d}{ds_0}u'_{sn} - \kappa_0(s_0)u'_{s\tau}\right\}$$

$$+i\frac{\gamma_1}{2\mu}\frac{d}{ds_0}\left\{\frac{d}{ds_0}u'_{sn} - \kappa_0(s_0)u'_{s\tau}\right\}.$$

It follows that

$$u'_{sn} \text{ and } \sigma_n$$

are bounded, while

$$u'_{s\tau} \text{ and } \tau_n$$

may have logarithmic singularities.

Numerical methods and accuracy

Unknown functions $g'(s)$ and $q(s)$ were approximated by

- Fourier series;
- Spline collocation methods;
- Taylor series.

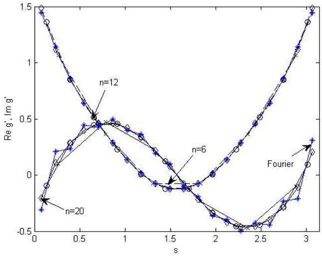
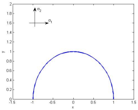
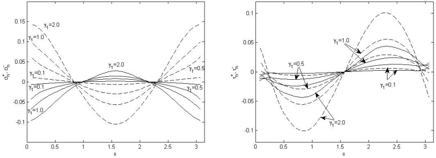


Figure : Convergence for Taylor and Fourier series approximations

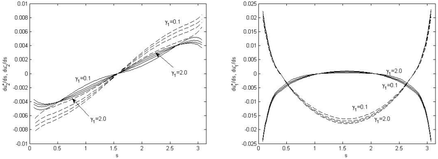
Semicircular crack under horizontal stretching at infinity



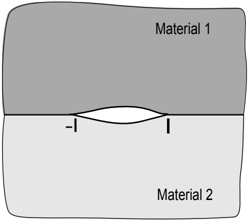
Normal and shear stresses.



Normal and tangential components of the derivative of displacements.



A straight interface crack (Zemlyanova, 2013)



Oscillating singularity: $\sigma + i\tau \sim Cr^{-1/2+i\gamma}$.

Alternative models:

- Contact zones (Comninou, 1977).
- No-slip zones (Mak et al, 1980).
- Intermediate layer (Atkinson, 1977).

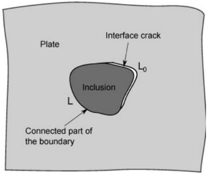


Figure : A partially debonded inclusion of arbitrary shape.

Jump momentum condition:

$$\text{grad}_{(\zeta)}\tilde{\gamma} + 2\tilde{\gamma}H\mathbf{n} + [[\ \mathbf{T}\ ]]\mathbf{n} = 0.$$

Surface tension $\tilde{\gamma}$ linearly depends on change in the mean curvature of the surface:

$$\tilde{\gamma} = \gamma_*(\text{div}_{(\zeta)}\mathbf{n} - \text{div}_{(\zeta_0)}\mathbf{n}_0).$$

Boundary conditions

$$(\sigma_n + i\tau_n)^{\pm}(s) = \frac{\gamma_*}{2\mu}\kappa_0(s)\left\{\frac{d}{ds}u'^{\pm}_{sn} - \kappa_0(s)u'^{\pm}_{s\tau}\right\}$$

$$+i\frac{\gamma_*}{2\mu}\frac{d}{ds}\left\{\frac{d}{ds}u'^{\pm}_{sn} - \kappa_0(s)u'^{\pm}_{s\tau}\right\}, \quad s \in [0, l_0],$$

$$(\sigma_n + i\tau_n)^{+}_{0}(s) - (\sigma_n + i\tau_n)^{-}(s) = \frac{\gamma^i}{2\mu}\kappa_0(s)\left\{\frac{d}{ds}u'^{\pm}_{sn} - \kappa_0(s)u'^{\pm}_{s\tau}\right\}$$

$$+i\frac{\gamma^i}{2\mu}\frac{d}{ds}\left\{\frac{d}{ds}u'^{\pm}_{sn} - \kappa_0(s)u'^{\pm}_{s\tau}\right\},, \quad s \in [l_0, L],$$

$$\frac{d}{dt}(u_1 + iu_2)^{+}_{0}(s) = \frac{d}{dt}(u_1 + iu_2)^{-}(s), \quad s \in [l_0, L],$$

where $\gamma_* = \gamma^+$ on L_0^+ and $\gamma_* = \gamma^-$ on L_0^-.

Mathematical techniques: Integral Representations

The complex potentials $\Phi_0(z)$ and $\Psi_0(z)$ for the inclusion can be written as:

$$\Phi_0(z) = \frac{1}{2\pi}\int_{L\cup L_0}\frac{g_0'(t)dt}{t-z} + \frac{(\kappa_0+1)^{-1}}{\pi i}\int_{L\cup L_0}\frac{q_0(t)dt}{t-z},$$

$$\Psi_0(z) = \frac{1}{2\pi}\int_{L\cup L_0}\left(\frac{\overline{g_0'(t)dt}}{t-z} - \frac{\bar{t}g_0'(t)dt}{(t-z)^2}\right) +$$

$$\frac{(\kappa_0+1)^{-1}}{\pi i}\int_{L\cup L_0}\left(\frac{\kappa_0\overline{q_0(t)dt}}{t-z} - \frac{\bar{t}q_0(t)dt}{(t-z)^2}\right),$$

The complex potentials $\Phi(z)$ and $\Psi(z)$ for the matrix can be written as:

$$\Phi(z) = \Gamma + \frac{1}{2\pi}\int_{L\cup L_0}\frac{g'(t)dt}{t-z} + \frac{(\kappa+1)^{-1}}{\pi i}\int_{L\cup L_0}\frac{q(t)dt}{t-z},$$

$$\Psi(z) = \Gamma' + \frac{1}{2\pi}\int_{L\cup L_0}\left(\frac{\overline{g'(t)dt}}{t-z} - \frac{\bar{t}g'(t)dt}{(t-z)^2}\right) +$$

$$\frac{(\kappa+1)^{-1}}{\pi i}\int_{L\cup L_0}\left(\frac{\kappa\overline{q(t)dt}}{t-z} - \frac{\bar{t}q(t)dt}{(t-z)^2}\right).$$

Extend the inclusion/outside matrix to the full complex plane assuming that the stresses and strains are equal to zero outside of the inclusion/matrix:

$$2q_0(t) = (\sigma_n + i\tau_n)_0^+(t) - (\sigma_n + i\tau_n)_0^-(t), \;\; t \in L \cup L_0,$$

$$i(\kappa_0 + 1)g_0'(t) = 2\mu_0 \frac{d}{dt}(u_1 + u_2)_0^+(t), \;\; t \in L \cup L_0,$$

$$2q(t) = (\sigma_n + i\tau_n)^+(t) - (\sigma_n + i\tau_n)^-(t), \;\; t \in L \cup L_0,$$

$$i(\kappa + 1)g'(t) = -2\mu \frac{d}{dt}(u_1 + u_2)^-(t), \;\; t \in L \cup L_0.$$

The problem can be reduced to the system of eight singular integro-differential equations which can be further regularized.

Numerical results

Method: Representations of unknown functions by Taylor polynomials.

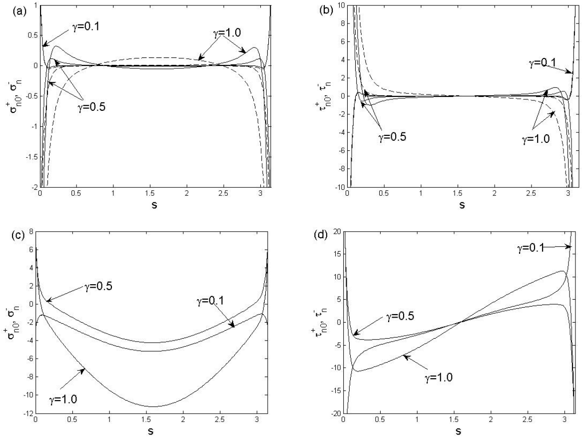


Figure : Graphs of the normal and shear stresses σ^+_{n0}, σ^-_n, τ^+_{n0}, τ^-_n on the curves L_0 and L

Comparison with known results

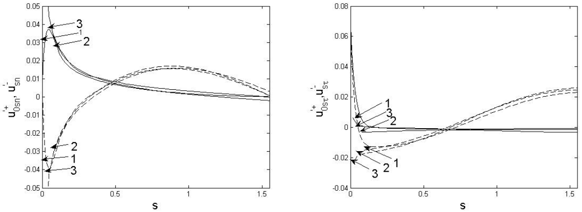


Figure : Graphs of the derivatives of the stresses $d(u_{1,2})_0^+/dt$, $du_{1,2}^-/dt$ on the fracture surface L_0

Mechanical parameters: $\mu_0 = \mu = 40$, $\nu_0 = \nu = 0.25$, $\gamma^+ = \gamma^- = 0.01$, $\gamma_i = 0.05$, $\sigma_1 = 1$, $\sigma_2 = 0$, $\alpha = 0$.

Advantages of the method

- Works for the fractures of arbitrary shape.
- Can be generalized for multiple fractures and multiple inclusions.

How to find the parameters?

Experimental results for tension loaded nanowires with rough surface:
“Atomistic elucidation of the effect of surface roughness on curvature-dependent surface energy, surface stress, and elasticity” by P. Mohammadi and P. Sharma.
Parameters in Steigmann-Ogden model have been computed.