The braided Hopf algebra structure of reflected Nichols algebras

Hans-Jürgen Schneider

LMU, München, Germany

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Joint work in progress with Istvan Heckenberger

The main problem

H Hopf algebra with bijective antipode, $\theta \geq 1$

Definition
$\mathcal{F}_\theta^H = \{M = (M_1, \dots, M_\theta) \mid M_1, \dots, M_\theta \text{ fin-dim. irr.} \in {}^H_H\mathcal{YD}\}$
$\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta), M \in \mathcal{F}_\theta^H$

$\mathcal{B}(M)$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$, $\deg(M_i) = \alpha_i$,
$\alpha_1, \dots, \alpha_\theta$ standard basis of $\mathbb{Z}^\theta$

What is the structure of $\mathcal{B}(M)$?

Find combinatorial invariants of $\mathcal{B}(M)$ (Weyl groupoid, root system ...)

Decomposition of $\mathcal{B}(M)$
"Lusztig's PBW-basis of $U_q(\mathfrak{g})^+$ for Nichols algebras"

Describe the right coideal subalgebras of $\mathcal{B}(M)$

Generalize $\mathcal{B}(M)$ to Nichols system for M

Part I: The reflection operator

Reflection of Yetter-Drinfeld modules

$$M \mapsto R_i(M) \in \mathcal{F}_\theta^H, \ 1 \leq i \leq \theta$$

Categorical interpretation of $\mathcal{B}(R_i(M))$ and generalization

1. The i-th reflection of M

Definition
M **admits the i-th reflection** if
$\forall j \neq i \ \exists - a_{ij}^M \in \mathbb{N}_0$ with

$$(\mathrm{ad} M_i)^{-a_{ij}^M}(M_j) \neq 0, \ \ (\mathrm{ad} M_i)^{1-a_{ij}^M}(M_j) = 0$$

Reflection operator
Assume that M admits the i-th reflection. Let

$$R_i(M) = (M_1', \ldots, M_\theta'),$$

where $M_i' = M_i^*$, $\forall j \neq i$, $M_j' = (\mathrm{ad} M_i)^{-a_{ij}^M}(M_j)$ in ${}_H^H\mathcal{YD}$.

$R_i(M) \in \mathcal{F}_\theta^H$ by [AHS 2010]

2. Hopf algebras with a projection in braided categories

$\mathcal{C} = \mathcal{C}(\otimes, I, c)$ (strict) monoidal braided category.
R Hopf algebra in $\mathcal{C}$, antipode of R an isomorphism.

(B, π, γ) **Hopf algebra triple over** $R \iff$
B Hopf algebra in $\mathcal{C}$, $\pi : B \to R, \gamma : R \to B$ Hopf algebra homomorphisms in $\mathcal{C}$ with $\pi\gamma = \mathrm{id}_R$

$$\begin{array}{ccccc} & & & & R \\ & & & \swarrow^{\gamma} & \downarrow {=} \\ B^{\mathrm{co}R} = K & \xrightarrow{\subseteq} & B & \xrightarrow{\pi} & R \end{array}$$

$\iff$ K Hopf algebra in ${}^R_R\mathcal{YD}(\mathcal{C})$, $K\#R \cong B$
(Radford-Majid bosonization)

Note: $\mathcal{C} = {}^H_H\mathcal{YD} \Rightarrow {}^R_R\mathcal{YD}(\mathcal{C})$ is naturally identified with ${}^{R\#H}_{R\#H}\mathcal{YD}$

3. Pre-Nichols and Nichols systems

S Hopf algebra in ${}^H_H\mathcal{YD}$,
$N_1, \dots, N_\theta \subseteq S$ fin.-dim. subobjects in ${}^H_H\mathcal{YD}$, $N = (N_1, \dots, N_\theta)$,
$f = (f_j)_{1 \leq j \leq \theta} : N \cong M$ in $\mathcal{F}^H_\theta$.

Definition
$\mathcal{N} = \mathcal{N}(S, N, f)$ is called a **pre-Nichols system for** M if

(1) S is generated as an algebra by $N_1, \dots, N_\theta$,

(2) $\sum_{j=1}^\theta N_j = \bigoplus_{j=1}^\theta N_j$, and

(3) S is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with grading given by $\deg(N_j) = \alpha_j$ for all $1 \leq j \leq \theta$.

$$p^{\mathcal{N}} : S \to \mathcal{B}(M) \quad \mathbb{N}_0^\theta\text{-graded Hopf algebra map,}$$

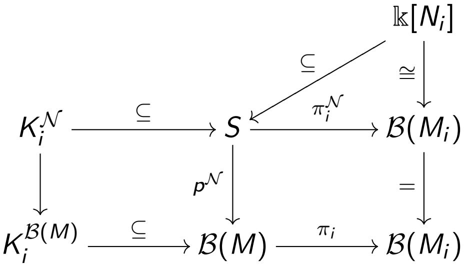
the **canonical map of** $\mathcal{N}$, defined by $f_j : N_j \xrightarrow{\cong} M_j \subseteq \mathcal{B}(M)$.

Definition

A pre-Nichols system $\mathcal{N} = \mathcal{N}(S, N, f)$ for M is called a **Nichols system for** M, if $p^{\mathcal{N}}$ defines bijective maps

(1) $\Bbbk[N_j] \cong \mathcal{B}(M_j)$ for all $1 \leq j \leq \theta$, and

(2) $(\mathrm{ad}^S N_i)^n(N_j) \cong (\mathrm{ad}^{\mathcal{B}(M)} M_i)^n(M_j)$ for all $1 \leq i, j \leq \theta, j \neq i$, and $n \geq 0$.



$K_i^{\mathcal{B}(M)}$ is generated by $(\mathrm{ad} M_i)^n(M_j), j \neq i, n \geq 0$.

$K_i^{\mathcal{N}}$ is generated by $(\mathrm{ad}^S N_i)^n(N_j), j \neq i, n \geq 0$.

4. The category isomorphism

Let $\mathcal{C} = {}^H_H\mathcal{YD}$, $1 \leq i \leq \theta$

Theorem [HS 2013b]
There is a naturally defined braided monoidal isomorphism of categories

$$(\Omega_i, \omega_i) : {}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}},$$

where $\Omega_i(V) = V$ as a Yetter-Drinfeld module over H, $\Omega_i(f) = f$ for morphisms f

$X \in \mathcal{YD}_{\mathrm{rat}}$ means: $\forall x \in X\ \exists n_0$ with $\mathcal{B}^n(M_i)x = 0$ (resp. $\mathcal{B}^n(M_i^*)x = 0$) $\forall n \geq n_0$

The basic construction

K Hopf algebra in ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$

$B := K\#\mathcal{B}(M_i) \mapsto \widetilde{B} := \Omega_i(K)\#\mathcal{B}(M_i^*)$ (Hopf algebras in ${}^H_H\mathcal{YD}$)

Theorem

Assume that M admits the i-th reflection, let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system for M. Let $\widetilde{N}_i = M_i^*$, and

$$\widetilde{S} = \Omega_i(K_i^{\mathcal{N}})\#\mathcal{B}(M_i^*),\ \widetilde{N}_j = (\mathrm{ad}^S N_i)^{-a_{ij}^M}(N_j), j \neq i.$$

(1) *For all $j \neq i$ and $n \geq 0$,*

$$(M_i^*)^n \cdot_\Omega \widetilde{N}_j = \begin{cases} (\mathrm{ad}^S N_i)^{-a_{ij}^M - n}(N_j), & \text{if } 0 \leq n \leq -a_{ij}^M \\ 0, & \text{if } n > -a_{ij}^M. \end{cases}$$

(2) *$\widetilde{S}$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}_H^H\mathcal{YD}$ with*

$$\deg(x \otimes y) = s_i^M(\deg^S(x) + \deg(y))$$

for homogeneous el. $x \in K_i^{\mathcal{N}}$ and $y \in \mathcal{B}(M_i^*)$, where $\deg(M_i^*) = -\alpha_i$. *For all* $1 \leq j \leq \theta$, $\deg(\widetilde{M}_j) = \alpha_j$.

On the proof

For all $j \neq i$, $p^{\mathcal{N}}$ induces an isomorphism between

$$(\mathrm{ad}^S \Bbbk[N_i](N_j) = N_j \oplus (\mathrm{ad}^S N_i)(N_j) \oplus \cdots \oplus (\mathrm{ad}^S N_i)^{-a_{ij}^M}(N_j), \text{ and}$$

$$(\mathrm{ad}\mathcal{B}(M_i))(M_j) = M_j \oplus (\mathrm{ad} M_i)(M_j) \oplus \cdots \oplus (\mathrm{ad} M_i)^{-a_{ij}^M}(M_j)$$

M_j is irreducible in ${}^H_H\mathcal{YD}$ $\Rightarrow$
$(\mathrm{ad}\mathcal{B}(M_i))(M_j)$ irreducible in ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ by [AHS 2010], hence $\Omega_i((\mathrm{ad}^S \Bbbk[N_i])(N_j))$ is irreducible.

Use structure of $\mathbb{Z}$-graded irreducible Yetter-Drinfeld modules.

Definition

Assume M admits the i-th reflection, $\mathcal{N} = \mathcal{N}(S, N, f)$ a Nichols system for M.

(1) Let $R_i(\mathcal{N}) := \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f})$, where $\widetilde{S}, \widetilde{N}$ are as in the theorem, $\widetilde{f}_j : \widetilde{N}_j \to R_i(M)_j, j \neq i$, is induced by $p^{\mathcal{N}}$, and $\widetilde{f}_i = \mathrm{id}$.

(2) We say that $\mathcal{N}$ admits the i-th reflection, if $R_i(\mathcal{N})$ is a Nichols system.

Note: $R_i(\mathcal{N})$ is a pre-Nichols system by the theorem.

Theorem[HS 2013a]

There is an isomorphism of Hopf algebras in ${}^H_H\mathcal{YD}$

$$\Phi : \mathcal{B}(R_i(M)) \to \Omega_i(K_i^{\mathcal{B}(M)})\#\mathcal{B}(M_i^*)$$

which is the identity on the components of $R_i(M)$.

5. The Hopf algebra structure after reflection

Theorem

K Hopf algebra in ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, $\widetilde{B} = \Omega_i(K)\#\mathcal{B}(M_i^*)$,

$$\begin{array}{ccccc} & & & & \mathcal{B}(M_i^*)^{\mathrm{cop}} \\ & & & \swarrow^{\subseteq} & \downarrow = \\ L = \left(\widetilde{B}^{\mathrm{cop}}\right)^{\mathrm{co}} & \xrightarrow{\subseteq} & \widetilde{B}^{\mathrm{cop}} & \xrightarrow{\widetilde{\pi}_i} & \mathcal{B}(M_i^*)^{\mathrm{cop}} \end{array}$$

Then $L := \mathcal{S}_{\widetilde{B}}^{-1}(\Omega_i(K))$, and

$$\varphi : \mathbf{K}^{\mathrm{cop}} \xrightarrow{\cong} \mathbf{L}, \ \mathbf{x} \mapsto \mathcal{S}_{\widetilde{\mathbf{B}}}^{-1}\mathcal{S}_{\mathbf{K}}(\mathbf{x}).$$

is an isomorphism of algebras and coalgebras in $\mathcal{C}$.

What we have done so far

K Hopf algebra in ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$

$B := K\#\mathcal{B}(M_i) \mapsto \widetilde{B} = \Omega_i(K)\#\mathcal{B}(M_i^*)$ **new Hopf algebra in** ${}^H_H\mathcal{YD}$

This is interesting, since if $B = \mathcal{B}(M)$, then $\widetilde{B} \cong \mathcal{B}(R_i(M))$.
On the other hand,

$$\widetilde{B}^{\mathrm{cop}} \cong L\#\mathcal{B}(M_i^*)^{\mathrm{cop}},\ \varphi : K^{\mathrm{cop}} \xrightarrow{\cong} L \text{ as algebras and coalgebras}$$

$\Longrightarrow B$ **and** $\widetilde{B}$ **are closely related**.

We will see that φ "is" the part of the Lusztig automorphism used to construct the higher root vectors of $U_q(\mathfrak{g})^+$

Part II: PBW-like decomposition and Cartan graph of Nichols algebras

Axioms for combinatorial structures (Weyl groupoid, real roots,...)

Stepwise construction of right coideal subalgebras, and of the "higher root vectors" using φ as an essential tool

Applications when there are only finitely many real roots:

Nichols algebra versions of Lusztig's PBW-basis

When Nichols systems are Nichols algebras

1. Semi-Cartan, Cartan graph and Weyl groupoid (axioms)

Rough idea: Generalize the root system of a Cartan matrix to a family of Cartan matrices

$\theta \geq 1$, $\mathcal{X}$ a non-empty set, $r_i : \mathcal{X} \to \mathcal{X}, 1 \leq i \leq \theta$, maps,
$A^X = (a^X_{jk})_{1 \leq j,k \leq \theta}, X \in \mathcal{X}$, generalized Cartan matrices
$s^X_i \in \mathrm{Aut}(\mathbb{Z}^\theta)$ with

$$s^X_i(\alpha_j) = \alpha_j - a^X_{ij}\alpha_i \qquad \text{for all } 1 \leq j \leq \theta$$

Definition
$\mathcal{G} = \mathcal{G}(\theta, \mathcal{X}, (r_i)_{1 \leq i \leq \theta}, (A^X)_{X \in \mathcal{X}})$ is called a **semi-Cartan graph** if

(1) $r_i^2 = \mathrm{id}$ for all i,

(2) $a^X_{ij} = a^{r_i(X)}_{ij}$ for all X and i, j.

Definition

Weyl groupoid $\mathcal{W}(\mathcal{G})$ of a semi-Cartan graph $\mathcal{G}$

category with objects $X \in \mathcal{X}$, morphisms generated by all

$X \xrightarrow{s_i^X} r_i(X)$

Formally $\mathrm{Hom}(X, Y)$ consists of the triples (Y, s, X) with

$Y = r_{i_m} \cdots r_{i_2} r_{i_1}(X), \; s = s_{i_1}^{r_{i_1}(X)} \cdots s_{i_m}^{r_{i_m} \cdots r_{i_2} r_{i_1}(X)} \in \mathrm{Aut}(\mathbb{Z})$

$$\begin{aligned}(Y = r_{i_m} \cdots r_{i_2} r_{i_1}(X) \quad &\xrightarrow{s_{i_m}^{r_{i_m} \cdots r_{i_2} r_{i_1}(X)}} \cdots \to r_{i_1}(X) \xrightarrow{s_{i_1}^{r_{i_1}(X)}} X) \\ &= (Y \xrightarrow{s} X) \\ &= \mathrm{id}_X s_{i_1} \cdots s_{i_m},\end{aligned}$$

$m \in \mathbb{N}_0, 1 \leq i_1, \ldots, i_m \leq \theta.$

The inverse of $s_i^X : X \to r_i(X)$ is $s_i^{r_i(X)} : r_i(X) \to X$.

$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{w(\alpha_i) \in \mathbb{Z}^\theta \mid w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X), 1 \leq i \leq \theta\}$ **real roots**

$\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \mathbb{N}_0^\theta$ **positive** real roots
$\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap -\mathbb{N}_0^\theta$ **negative** real roots.
For any $X \in \mathcal{X}$ and $1 \leq i, j \leq \theta$ let

$$m_{ij}^X = |\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|.$$

Definition

The semi-Cartan graph $\mathcal{G}$ is a **Cartan graph** if

(1) $\forall\, X \in \mathcal{X}$, $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ consists of positive and negative roots.

(2) $\forall\, X \in \mathcal{X}, 1 \leq i, j \leq \theta$, $(r_i r_j)^{m_{ij}^X}(X) = X$, if $m_{ij}^X < \infty$.

The Weyl groupoid of a Cartan graph is a Coxeter groupoid (Heckenberger, Yamane 2008)

Example

Let $\theta = 2$, $\mathcal{X} = \{X_1, X_2\}$, $r_1 : \mathcal{X} \to \mathcal{X}$ the non-trivial permutation and $r_2 : \mathcal{X} \to \mathcal{X}$ the identity. Let

$$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{X_2} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Then $\mathcal{G} = \mathcal{G}(2, \mathcal{X}, r, A)$ is a semi-Cartan graph. The Weyl groupoid is generated by the morphisms

$$s = s_1^{X_2} : X_2 \to X_1,\ t = s_1^{X_1} : X_1 \to X_2,$$
$$u = s_2^{X_1} : X_1 \to X_1,\ v = s_2^{X_2} : X_2 \to X_2.$$

Then s and t are inverse isomorphisms in $\mathcal{W}(\mathcal{G})$, and u, v are self-inverse.

$\mathrm{Aut}(X_1)$ is generated by u and $(X_1 \xrightarrow{t} X_2 \xrightarrow{v} X_2 \xrightarrow{s} X_1)$.

$$\boldsymbol{\Delta}^{X_1\,\mathrm{re}} = \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 1^2 2^3, \pm 1^3 2^4, \pm 1^3 2^5\},$$
$$\boldsymbol{\Delta}^{X_2\,\mathrm{re}} = \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 12^4, \pm 1^2 2^3, \pm 1^2 2^5\},$$
$$(a\alpha_1 + b\alpha_2 = 1^a 2^b,\ a, b \in \mathbb{Z}.)$$

2. Semi-Cartan and Cartan graph of $\mathcal{B}(M)$

Assume that M admits all reflections $R_{i_1} \cdots R_{i_n}(M), n \geq 0$.

(This is true if $\mathcal{B}(M)$ is fin.-dim., or if GK-dim($\mathcal{B}(M)$) is finite, and all tensor powers of $M_1 \oplus \cdots \oplus M_\theta$ are semisimple in ${}_H^H\mathcal{YD}$ [HS 2010].) Let

$$\mathcal{F}_\theta^H(M) = \{R_{i_1} \cdots R_{i_n}(M) \mid n \in \mathbb{N}_0, i_1, \ldots, i_n \in I\}$$

$$\mathcal{X}(M) = \{[N] \mid N \in \mathcal{F}_\theta^H(M)\} \text{ isomorphism classes}$$

$$r_i : \mathcal{X}(M) \to \mathcal{X}(M), r_i([N]) = [R_i(N)], A^{[N]} = (a_{ij}^N)_{1 \leq i,j \leq \theta} \; (a_{ii}^N = 2)$$

Theorem

(1) *$\mathcal{G}(M) = \mathcal{G}(\theta, \mathcal{X}(M), (r_i)_{1 \leq i \leq \theta}, (A^X)_{X \in \mathcal{X}(M)})$ is a semi-Cartan graph. [AHS 2010]*

(2) *Assume $\mathbf{\Delta}^{[M]\,\mathrm{re}}$ is finite. Then $\mathcal{G}(M)$ is a Cartan graph. [HS 2010], [HS 2013a]*

$$\mathcal{W}(M) = \mathcal{W}(\mathcal{G}(M))$$

3. Stepwise construction of right coideal subalgebras, and of the "higher root vectors"

Let $1 \leq i \leq \theta$, and assume that M admits the i-th reflection. Recall

$$\Phi : \mathcal{B}(R_i(M)) \xrightarrow{\cong} \Omega_i(K_i^{\mathcal{B}(M)}) \# \mathcal{B}(M_i^*).$$

Definition

$$T_i^{\mathcal{B}(R_i(M))} : L_i^{\mathcal{B}(R_i(M))} \xrightarrow{\Phi} {}^{\mathrm{co}\mathcal{B}(M_i^*)}(\Omega_i(K_i^{\mathcal{B}(M)}) \# \mathcal{B}(M_i^*)) \xrightarrow{\varphi^{-1}} K_i^{\mathcal{B}(M)}$$

The T_i play the role of Lusztig's T_i.

Definition

Let

$$\mathcal{K}(\mathcal{B}(M)) = \{E \mid E \subseteq \mathcal{B}(M) \ \mathbb{N}_0^\theta\text{-gr. right coid. subalg. in } {}^H_H\mathcal{YD}\},$$
$$\mathcal{K}_i^+(\mathcal{B}(M)) = \{E \mid E \in \mathcal{K}(\mathcal{B}(M)), M_i \subseteq E\},$$
$$\mathcal{K}_i^-(\mathcal{B}(M)) = \{E \mid E \in \mathcal{K}(\mathcal{B}(M)), M_i \nsubseteq E\}.$$

Lemma

$\mathcal{K}_i^-(\mathcal{B}(M)) = \{E \mid E \in \mathcal{K}(\mathcal{B}(M)), E \subseteq L_i^{\mathcal{B}(M)}\}.$

The fundamental relation between right coideal subalgebras in $\mathcal{B}(M)$ and in $\mathcal{B}(R_i(M))$.

Theorem

$1 \leq i \leq \theta$, $M \in \mathcal{F}_\theta^H$ *admits the i-th reflection.*

(1) *The map*

$$t_i^{\mathcal{B}(R_i(M))} : \mathcal{K}_i^-(\mathcal{B}(R_i(M))) \to \mathcal{K}_i^+(\mathcal{B}(M)),$$
$$E \mapsto T_i^{\mathcal{B}(R_i(M))}(E)\Bbbk[M_i],$$

is bijective with inverse $E \mapsto (T_i^{\mathcal{B}(R_i(M))})^{-1}(E \cap K_i^{\mathcal{B}(M)})$.

(2) *The multiplication map* $T_i^{\mathcal{B}(R_i(M))}(E) \otimes \Bbbk[M_i] \to T_i^{\mathcal{B}(R_i(M))}(E)\Bbbk[M_i]$ *is bijective for all* $E \in \mathcal{K}_i^-(\mathcal{B}(R_i(M)))$.

Stepwise construction

Let $1 \leq k \leq l, 1 \leq i_1 \dots, i_l \leq \theta$. Consider in $\mathcal{W}(M)$,

$$[\mathbf{R_{i_l}} \cdots \mathbf{R_{i_1}}(\mathbf{M})] \xrightarrow{\mathbf{s_{i_l}}} [\mathbf{R_{i_{l-1}}} \cdots \mathbf{R_{i_1}}(\mathbf{M})] \cdots \xrightarrow{\mathbf{s_{i_2}}} [\mathbf{R_{i_1}}(\mathbf{M})] \xrightarrow{\mathbf{s_{i_1}}} [\mathbf{M}]$$

Let $M_1^{\mathcal{B}(M)}(i_1, \dots, i_l) = M_{i_1}$. We say that

$$M_k^{\mathcal{B}(M)}(i_1, \dots, i_l) = T_{i_1} \cdots T_{i_{k-1}}((R_{i_{k-1}} \cdots R_{i_1}(M))_{i_k}), 2 \leq k \leq l,$$
$$E^{\mathcal{B}(M)}(i_1, \dots, i_l) = t_{i_1} \cdots t_{i_l}(\Bbbk 1)$$

are **well-defined**, where

$$T_{i_p} = T_{i_p}^{\mathcal{B}(R_{i_p} \cdots R_{i_1}(M))} : L_{i_p}^{\mathcal{B}(R_{i_p} \cdots R_{i_1}(M))} \to K_{i_p}^{\mathcal{B}(R_{i_{p-1}} \cdots R_{i_1}(M))}$$
$$t_{i_p} = t_{i_p}^{\mathcal{B}(R_{i_p} \cdots R_{i_1}(M))} : \mathcal{K}_{i_p}^{-}(\mathcal{B}(R_{i_p} \cdots R_{i_1}(M))) \to \mathcal{K}_{i_p}^{+}(\mathcal{B}(R_{i_{p-1}} \cdots R_{i_1}(M))),$$

if all the compositions are well-defined.

Theorem

$M \in \mathcal{F}_\theta^H$ admits all reflections, $l \geq 1$, and $1 \leq i_1, \dots, i_l \leq \theta$. Assume that $(i_1, \dots, i_l)$ is $[M]$-reduced, that is, $\beta_1, \dots, \beta_l$ are pairwise distinct elements in $\mathbb{N}_0^\theta$, where $\beta_k = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then the $M_k^{\mathcal{B}(M)}(i_1, \dots, i_l) \subseteq \mathcal{B}(M)$, and $E^{\mathcal{B}(M)}(i_1, \dots, i_l) \subseteq \mathcal{B}(M)$ are well-defined. For all $1 \leq k \leq l$, let

$$M_{\beta_k} = M_k^{\mathcal{B}(M)}(i_1, \dots, i_l).$$

Then

(1) *For all $1 \leq k \leq l$, $M_{\beta_k} \subseteq E^{\mathcal{B}(M))}(i_1, \dots, i_l)$ is a finite-dimensional irreducible subobject in ${}_H^H\mathcal{YD}$ of degree β_k.*

(2) *For all $1 \leq k \leq l$, there is an isomorphism $\mathcal{B}(M_{\beta_k}) \cong \Bbbk[M_{\beta_k}]$*

(3) *The multiplication map $\Bbbk[M_{\beta_l}] \otimes \cdots \otimes \Bbbk[M_{\beta_1}] \to E^{\mathcal{B}(M)}(i_1, \dots, i_l)$ is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$.*

Notes on the proof

By induction on l

Works for Nichols systems $\mathcal{N} = \mathcal{N}(S, N, f)$:

$p^{\mathcal{N}}$ induces an isomorphism $E^{\mathcal{N}}(i_1, \ldots, i_l) \cong E^{\mathcal{B}(M)}(i_1, \ldots, i_l)$

$p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is an isomorphism, if $E^{\mathcal{N}}(i_1, \ldots, i_l) = \mathcal{B}(M)$

4. When the number of real roots is finite

Need finiteness assumption to actually reach $\mathcal{B}(M) = E^{\mathcal{B}(M)}(i_1, \ldots, i_m)$ for some sequence $(i_1, \ldots, i_m)$.
Get "root space decomposition" of $\mathcal{B}(M) \Rightarrow \mathcal{G}(M)$ is Cartan.

Theorem[HS 2013a]
Assume that M admits all reflections. The following are equivalent.

(1) *The length of all $[N]$-reduced sequences, for all $N \in \mathcal{F}_\theta^H(M)$, is bounded.*
(2) $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}$ *is finite.*

In this case, $\mathcal{G}(M)$ is a Cartan graph. If $w_0 = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_m}$ is the longest word, in $\mathrm{Hom}(\mathcal{W}(M), [M])$, then

$$\Bbbk[M_{\beta_m}] \otimes \cdots \otimes \Bbbk[M_{\beta_1}] \xrightarrow{\cong} E^{\mathcal{B}(M))}(i_1, \ldots, i_m) = \mathcal{B}(M).$$

Theorem

Assume M admits all reflections, and $\mathbf{\Delta}^{[M]\,\mathrm{re}}$ is finite. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system for M which admits all reflections. Then the canonical map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is an isomorphism.

Theorem

Let $\mathrm{char}(\Bbbk) = 0$*. Then any fin.-dim. pre-Nichols system for* M*, where all* M_i *are one-dim., is Nichols.*

Corollary(I. Angiono)

Let $\mathrm{char}(\Bbbk) = 0$ *and* H *a finite-dimensional pointed Hopf algebra with abelian group* $G(H)$*. Then* H *is generated by group-likes and skew-primitive elements.*

Corollary

Assume that M admits all reflections and $\mathcal{G}(M)$ is finite.

(1) $\boldsymbol{\Delta}_+^{[M]\,\mathrm{re}} = \{\beta_1, \dots, \beta_m\}$.

(2) *Let $P \in \mathcal{F}_\theta^H(M)$, and $1 \le i \le \theta$. Then $P_i \cong M_{\beta_k}$ or $P_i \cong M^*_{\beta_k}$ in ${}^H_H\mathcal{YD}$ for some $1 \le k \le m$.*

(3) *Let $1 \le i, j \le \theta, i \ne j$, and $0 \le t \le -a_{ij}^M$. Then for some $1 \le k \le m$,*

$$\alpha_j + t\alpha_i = \beta_k,\ (\mathrm{ad} M_i)^t(M_j) \cong M_{\beta_k} \text{ in } {}^H_H\mathcal{YD}.$$

In particular, $(\mathrm{ad} M_i)^t(M_j)$ **is irreducible in ${}^H_H\mathcal{YD}$.**

Corollary

Assume that M admits all reflections and $\mathcal{G}(M)$ is finite. For all $x \in M_{\beta_k}$, $1 \leq k \leq m$,

$$\Delta_{\mathcal{B}(M)}(x) \in x \otimes 1 + 1 \otimes x + \mathbb{k}[M_{\beta_{k-1}}] \cdots \mathbb{k}[M_{\beta_1}] \otimes \mathcal{B}(M).$$

Corollary

Assume that M admits all reflections and $\mathcal{G}(M)$ is finite. For all $1 \leq p < q \leq l, x \in M_{\beta_p}, y \in M_{\beta_q}$,

$$xy - (x_{(-1)} \cdot y)x_{(0)} \in \mathbb{k}[M_{\beta_{q-1}}]\mathbb{k}[M_{\beta_{q-2}}] \cdots \mathbb{k}[M_{\beta_{p+1}}].$$

As a special case, we obtain the commutation rule of Levendorskii and Soibelman for quantum groups without any case by case consideration.

Theorem[HS 2013a]
Assume that M admits all reflections and $\mathcal{G}(M)$ is finite. Then

$$\begin{aligned} \mathrm{Hom}(\mathcal{W}(M),[M]) \to \mathcal{K}(\mathcal{B}(M)),\ w \mapsto E^{\mathcal{B}(M)}(w), \\ E^{\mathcal{B}(M)}(w) = E^{\mathcal{B}(M)}(j_1,\dots,j_l), \\ w = \mathrm{id}_{[M]} s_{j_1}\cdots s_{j_l} \textit{ reduced}, \end{aligned}$$

is bijective.

5. Application to right coideal subalgebras in $U_q^{\geq 0}(\mathfrak{g})$

$U = U_q(\mathfrak{g})$, q not a root of unity, $\mathfrak{g}$ semisimple Lie algebra.
Let w be an element of the Weyl group W, and let $s_{i_1} \cdots s_{i_m}$ be a reduced decomposition of w. Recall (see Jantzen's book) that $U^+[w]$ is the linear span of the products

$$E_{\beta_m}^{a_m} \cdots E_{\beta_2}^{a_2} E_{\beta_1}^{a_1}, \quad a_1, \ldots, a_m \in \mathbb{N}_0,$$

where $E_{\beta_l} = T_{i_1} \cdots T_{i_{l-1}}(E_{i_l})$ for all $1 \leq l \leq m$.

Theorem[HS 2013a]
The map $w \mapsto U^+[w]U^0$ from W to the set of right coideal subalgebras of $U^{\geq 0}$ containing the group algebra U^0, is a bijection.

Conjecture of Kharchenko (2009) for simple $\mathfrak{g}$:
Number of right coideal subalgebras $= |W|$.

Theorem[HS 2013a]
If w_0 is the longest word, we obtain Lusztig's PBW-basis of U^+, $\mathbb{k}[M_{\beta_m}] \otimes \cdots \otimes \mathbb{k}[M_{\beta_1}] \xrightarrow{\cong} U^+$, $\mathbb{k}[M_{\beta_l}]$ *is a polynomial ring in* E_{β_l}.

The first proof without any case by case considerations, no checking of the braid relations.