Nichols Algebras and Their Interactions with Lie Theory, Hopf Algebras and Tensor Categories

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1 Overview of the Field

Nichols algebras are generalizations of symmetric algebras in the context of braided tensor categories. They are graded algebras with rigid properties that make them ubiquitous and interesting objects on their own. The first appearance was in a paper of Nichols in 1978 in the search of new examples of Hopf algebras. Rediscovered later by Woronowicz in non-commutative calculus, they provide also a conceptual explanation of the construction of quantum groups, in different guises (Lusztig, Rosso, Schauenburg). Nichols algebras are fundamental structural invariants of pointed Hopf algebras as highlighted in the Lifting Method (Andruskiewitsch-Schneider) and so it is expected that they play a significant role in any problem on such Hopf algebras.

2 Recent Developments and Open Problems

2.1 Hopf algebras with finite Gelfand-Kirillov dimension

The Gelfand-Kirillov dimension, $\mathrm{GK\text{-}dim}$ for short, is a sort of non-commutative version of the dimension. Since loosely speaking, Hopf algebras are the quantum analogues of groups, it is natural to ask for the quantum analogues of algebraic groups—and these are the Hopf algebras with finite $\mathrm{GK\text{-}dim}$, as this condition appears to behave better than noetherianity. See [19] and references therein.

2.1.1 Noetherianity versus finite Gelfand-Kirillov dimension

Most of the questions below are from [2, 3]. Let us consider the following conditions on a Hopf algebra H:

(1) H is affine (i.e., finitely generated as an algebra).

(2) H is noetherian.

(3) $\mathrm{GK\text{-}dim}\, H < \infty$.

Here are some well-known open questions on the relations among these conditions:

Question 1. Is a noetherian Hopf algebra necessarily affine? That is, does (2) imply (1)?

The answer is yes if H is commutative (Molnar, 1975). This was also asked assuming that the Hopf algebra is PI [35]. Now (3) does not imply (2): take for instance $\Lambda(V)\#\Bbbk\mathbb{Z}/2$, with V infinite-dimensional.

Example 1. Let G be a group. Then the group algebra $H = \Bbbk G$ has finite GK-dim iff G is nilpotent-by-finite (Gromov's theorem). Also if G is polycyclic-by-finite, then H is noetherian. Hence (2) does not imply (3).

Question 2. When is a group algebra $H = \Bbbk G$ noetherian? Is G necessarily polycyclic-by-finite?

Question 3. Is an affine Hopf algebra with finite $\operatorname{GK-dim}$ necessarily noetherian? That is, do (1) and (3) imply (2)? At least if H is a domain?

The examples of Hopf algebras with non-bijective antipode are huge. Thus one may expect that growth conditions would imply bijectivity. Indeed, as a consequence of [34, Theorem A], we have:

Theorem 1. *The antipode of a Hopf algebra domain with finite* $\operatorname{GK-dim}$ *is bijective.*

However, the following is still open:

Conjecture 1. (Skryabin). The antipode of a noetherian Hopf algebra is bijective.

We assume from now on that the antipode is bijective. Let us focus on $\operatorname{GK-dim}$.

Question 4. If H is a Hopf algebra, is $\operatorname{GK-dim} H \in \mathbb{Z} \cup \{\infty\}$? At least if H is pointed?

Question 5. Let H be a Hopf algebra domain with finite $\operatorname{GK-dim}$. Is $\dim_k H$ countable? Is H is a filtered union of affine Hopf subalgebras?

2.1.2 The Lifting method

Let H be a pointed Hopf algebra with group G of grouplikes. Let $\operatorname{gr} H$ be the graded Hopf algebra associated to its coradical filtration, so that $\operatorname{gr} H \simeq R\#\Bbbk G$, where $R = \oplus_{n\geq 0} R^n$ is a coradically graded connected Hopf algebra in the category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of Yetter-Drinfeld modules over the group algebra $\Bbbk G$. Let $V = R^1$; this important invariant of H is named its *infinitesimal braiding*. Let us consider the following conditions on H:

(1) $\operatorname{GK-dim} H < \infty$.

(2) $\operatorname{GK-dim} \operatorname{gr} H < \infty$.

(3) $\operatorname{GK-dim} R < \infty$ and $\operatorname{GK-dim} \Bbbk G < \infty$.

By general reasons, $\operatorname{GK-dim} \operatorname{gr} H \leq \operatorname{GK-dim} H$, i. e. (1) implies (2) while the converse holds if $\operatorname{gr} H$ is finitely generated, see [27, 6.5, 6.6] (in particular this also applies if the coradical of H is a Hopf subalgebra, a setting where the preceding considerations still make sense). Clearly (2) implies (3).

Theorem 2. [36] *If* $\dim V < \infty$*, then* $\operatorname{GK-dim} R + \operatorname{GK-dim} \Bbbk G = \operatorname{GK-dim} \operatorname{gr} H$*; when R is affine,* $\operatorname{GK-dim} \operatorname{gr} H = \operatorname{GK-dim} H$*. Thus, if R is affine, then* (1) $\iff$ (2) $\iff$ (3)*.*

The argument in [36] applies more generally when the coradical of H is a Hopf subalgebra.

Example 2. Let $G = \mathbb{Z}^2$, $g = (1, 0)$, $h = (0, 1)$. Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with linear basis $(v_i)_{i\in\mathbb{Z}}$, grading $V = V_g$, and action $g \cdot v_i = -v_i$, $h \cdot v_i = v_{i+1}$, $i \in \mathbb{Z}$. Let $H = \Lambda(V)\#\Bbbk G \simeq \operatorname{gr} H$; here $R = \mathcal{B}(V) = \Lambda(V)$. Then $\operatorname{gr} H = \Bbbk\langle v_0, g, h\rangle$ is affine, but R is not; $\operatorname{GK-dim} H = \infty > 2 = \operatorname{GK-dim} \mathcal{B}(V) + \operatorname{GK-dim} \Bbbk G$. The Hopf subalgebra $R\#\Bbbk\langle g\rangle$ has $\operatorname{GK-dim} = 1$ but is neither affine nor noetherian.

Question 6. (Assume that $\operatorname{char} \Bbbk = 0$). Let H be a pointed affine Hopf algebra with finite $\operatorname{GK-dim}$. Is $\dim V < \infty$? Is $\operatorname{gr} H$ necessarily affine?

The subalgebra of R generated by V is the Nichols algebra of V. Thus, it is natural to ask:

Question 7. What are all braided vector spaces V such that $\text{GK-dim}\,\mathcal{B}(V) < \infty$?

The cases where $\dim V < \infty$ and/or $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, G nilpotent-by-finite, deserve primary attention.

Example 3. V is of diagonal type if it has a basis $(x_i)_{1\le i\le\theta}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, for all i and j, where $(q_{ij})_{1\le i,j\le\theta}$ is a matrix with non-zero entries and $q_{ii} \neq 1$ for all i. Furthermore, V is generic if q_{ij} is either 1 or not a root of 1, for all i and j. In this case, $\text{GK-dim}\,\mathcal{B}(V) < \infty$ if and only if (q_{ij}) is of finite Cartan type (Andruskiewitsch–Angiono, Rosso). This leads to the classification of pointed Hopf algebras that are domains with finite GK-dim and generic infinitesimal braiding (Andruskiewitsch–Schneider).

If V is of diagonal type, then $\mathcal{B}(V)$ bears a combinatorial structure—a (generalized) root system, GRS for short. Nichols algebras of diagonal type with finite GRS were classified in [21].

A complete answer to Question 7 would be just a partial step towards understanding pointed Hopf algebras with finite GK-dim: there may be infinitely many H with finite GK-dim containing $\mathcal{B}(V)$.

2.2 Homological aspects of Hopf algebras

2.2.1 Regularity

A noetherian algebra A provided with an augmentation $\varepsilon : A \to \Bbbk$ is *AS-Gorenstein* if

$$\operatorname{injdim} A_A = \operatorname{injdim} {}_A A = d \in \mathbb{N}, \qquad \dim \operatorname{Ext}^i_A({}_A\Bbbk, {}_AA) = \dim \operatorname{Ext}^i_A(\Bbbk_A, A_A) = \delta_{i,d}.$$

If, further, $\operatorname{gldim} A = d$ and $\text{GK-dim}\, A < \infty$, then A is *AS-regular* of dimension d. (There are graded versions of these notions for A $\mathbb{N}_0$-graded connected). It seems that there is no counterexample to the following:

Question 8. (Brown-Goodearl). Is every noetherian affine Hopf algebra AS-Gorenstein?

For instance, a finite-dimensional Hopf is AS-Gorenstein with $d = 0$. See [4, 35] for other examples. On the other hand, a finite-dimensional Hopf algebra is AS-regular if and only if it is semisimple. Also, commutative affine Hopf algebras (in characteristic 0), enveloping algebras of finite-dimensional Lie algebras, quantised enveloping algebras and quantised function algebras are AS-regular. However

- If G is polycyclic-by-finite, then $\Bbbk G$ is AS-regular if and only if G has no element of order $p = \operatorname{char} \Bbbk$.
- (Goodearl-Zhang). There is a family of noetherian Hopf algebra domains of GK-dim 2 which are AS-Gorenstein, but not AS-regular.

Question 9. Characterize (noetherian, or finite GK-dimensional) (affine) Hopf algebras that are AS-regular.

2.2.2 Homological integral

Let H be an AS-Gorenstein Hopf algebra with $\operatorname{injdim} H = d$. The (left) homological integral $\int_H^\ell$ of H is the 1-dimensional H-bimodule $\operatorname{Ext}^d_H({}_H\Bbbk, {}_HH)$ (hence the usual integral is recovered when $\dim H < \infty$) [28]. Let χ be the character of the right structure on $\int_H^\ell$.

Theorem 3. [4] *H has a rigid dualising complex $R \simeq {}^{\nu}H^1[d]$. Furthermore, the Nakayama automorphism ν equals $\mathcal{S}^2\tau^\ell_\chi$, where τ^ℓ_χ denotes the left winding automorphism of H from χ.*

A first consequence of this theorem is a generalization of the celebrated Radford's formula for $\mathcal{S}^4$.

Theorem 4. [4] *$\mathcal{S}^4 = \gamma \circ \tau^\ell_{-\chi} \circ \tau^r_\chi$ for some inner automorphism γ of H.*

Next, combining with results of Van den Bergh, one gets:

Theorem 5. [4] *If H is AS-regular and M is an H-bimodule, then $H^i(H, {}^{\nu}M^1) \simeq H^{d-i}(H, M)$ for all i.*

2.2.3 Homological aspects of Nichols algebras

Being on fundamental parts of the structure of pointed Hopf algebras, it is natural to approach the preceding questions by considering first their counterparts for Nichols algebras. Thus, let V be a braided vector space, $\dim V < \infty$; assume that $\mathcal{B}(V)$ is noetherian and that $\text{GK-dim}\,\mathcal{B}(V) < \infty$.

Question 10. If $\mathcal{B}(V)$ is a domain, is it AS-regular?

Question 11. Is $\mathcal{B}(V)$ AS-Gorenstein?

Question 12. What is the (homological) Nakayama automorphism of $\mathcal{B}(V)$?

Question 13. When has $\mathcal{B}(V)$ a classical ring of quotients?

Question 14. Is the Krull dimension of $\mathcal{B}(V)$ finite?

2.3 Quantum symmetries

A classical problem is the study of the action of a group G on a variety X. When X is affine, sensible information is provided by the corresponding ring of invariants. Here and in §3.2, the theme is a quantum version of this situation, with an algebra A (with favourable properties) instead of the variety X and a Hopf algebra H in the place of G, so that A is a left H-module algebra. To avoid trivialities, it is assumed that the action of H on A is *inner-faithful*, that is the only Hopf ideal I of H such that $IA = 0$ is $I = 0$.

Question 15. For suitable classes of algebras, what kind of Hopf algebras may act?

A first reduction is given by a result of

Theorem 6. *(Skryabin-Van Oystaeyen). Assume that H is finite-dimensional. If A has a right Artinian classical right quotient ring Q, then the action of H extends (uniquely) to Q.*

It is also worth mentioning:

Conjecture 2. (Artamonov). Any finite-dimensional Hopf algebra can act inner-faithfully on a quantum torus (hence on its division algebra of quotients).

The archetypical examples of A in the commutative setting are the polynomial algebras; good candidates to be their correct analogues in the non-commutative world are graded connected AS-regular algebras; these are classified in dimension ≤ 3 but the zoo is large in higher dimension. A suitable test subfamily is that of *quantum polynomial rings*, i.e. AS-regular graded domains of global dimension n such that $H_A(t) = (1-t)^{-n}$, where $H_V(t)$ denotes the Hilbert series of a (locally finite) graded vector space V.

Question 16. What properties might the ring of invariants A^H have, under assumptions on A and H?

Relevant results in classical invariant theory that may serve as roadmap: Noether's theorem on integrality over invariant subrings, the Shephard-Todd-Chevalley Theorem (a criterion for a fixed subring to be a polynomial ring), the Watanabe Theorem (a criterion for a fixed subring to be Gorenstein) and the Kac-Watanabe-Gordeev Theorem (a criterion for a fixed subring to be a complete intersection ring).

A generalization of Noether's theorem was achieved by Etingof [12] assuming suitable conditions (building on previous work by Skryabin in the commutative case). Also, the Watanabe Theorem was extended by Jørgensen and Zhang through their notion of homological determinant, defined in terms of local cohomology.

Theorem 7. *(Jørgensen-Zhang) If G is a finite group of graded automorphisms of an AS-regular algebra A with $\operatorname{hdet}(g) = 1$ for all $g \in G$, then A^G is AS-Gorenstein.*

2.4 Fusion categories

All known fusion categories in characteristic 0 are of two sorts: either weakly group-theoretical, or else a category $\mathcal{C}(\mathfrak{g}, q)$, where $\mathfrak{g}$ is a simple Lie algebra and q is a root of unity (of suitable order).

Let G be a finite group and $\omega \in Z^3(G, \Bbbk^\times)$. The tensor category of finite-dimensional G-graded vector spaces with associativity constraint given by ω is denoted Vec_G^ω. If $F < G$ and α is a 2-cochain on F such that $d\alpha = \omega_{|F\times F\times F}$, then the twisted group algebra $\Bbbk_\alpha F$ is an associative algebra in Vec_G^ω. The category $\mathcal{C}(G, \omega; F, \alpha)$ of $\Bbbk_\alpha F$-bimodules in Vec_G^ω is a fusion category; fusion categories like this are called *group-theoretical*. There is a larger class of *weakly group-theoretical* fusion categories with a more complicated definition, see [15]. As a matter of fact, the representation category of every known semisimple Hopf algebra H is weakly group-theoretical.

A spherical category $\mathcal{C}$ gives rise to a factor category $\underline{\mathcal{C}}$, with objects as $\mathcal{C}$ and the spaces of morphisms are $\mathrm{Hom}_{\underline{\mathcal{C}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{J}(X, Y)$, $X, Y \in \mathcal{C}$, where $\mathcal{J}(X, Y)$ is the space of degenerate morphisms (in terms of a trace defined by the pivotal element of $\mathcal{C}$) (Barrett and Westbury). Particularly, if $\mathcal{C} = \mathrm{Rep}\, H$, where H is a spherical Hopf algebra, then $\underline{\mathrm{Rep}\, H} = \underline{\mathcal{C}}$ is a semisimple spherical tensor category, whose irreducible objects are the indecomposable H-modules with non-zero quantum dimension. For further use, let $\mathrm{Indec}\, \mathcal{C}$, respectively $\mathrm{Irr}\, \mathcal{C}$, be the set of isomorphism classes of indecomposable, respectively irreducible or simple, objects in an additive category $\mathcal{C}$. Two relevant examples are:

- (S. Gelfand–Kazhdan, Georgiev–Mathieu). Let G be a simple algebraic group over a field $\Bbbk$ of characteristic $p >$ the Coxeter number of G and $\mathcal{C} = \mathrm{Rep}\, G$. Then the subcategory $\mathcal{T}$ of $\mathcal{C}$ consisting of tilting modules gives rise to a fusion subcategory of $\underline{\mathcal{C}}$.
- (Andersen). An analogous example in characteristic 0: take $\mathcal{C}$ the category of type I representations of a quantum group at a root of unity (of suitable order). Again the subcategory $\mathcal{T}$ of $\mathcal{C}$ consisting of tilting modules gives rise to a fusion subcategory of $\underline{\mathcal{C}}$.

Question 17. What are the fusion categories that can not be obtained from finite groups and quantum groups?

One approach to this question is through the Witt group [9, 11]. Another is in the context of subfactors taking as primary invariant the Jones index. Progress in this line was surveyed by Noah Snyder in his talk, including a discussion of the so-called Haagerup exotic fusion categories [30].

3 Presentation Highlights

3.1 Hopf algebras with finite Gelfand-Kirillov dimension

3.1.1 Connected Hopf algebras

A Hopf algebra is *connected* if its coradical has dimension 1, i.e. pointed with trivial group of group-likes.

Theorem 8. [13] *A connected Hopf algebra with finite* $\mathrm{GK\text{-}dim}$ *is a quantum deformation of the algebra of rational functions on a nilpotent algebraic group.*

In fact, [13] applies also without the finiteness of $\mathrm{GK\text{-}dim}$ hypothesis, with proalgebraic instead of algebraic in the conclusion. Theorem 8 reduces the classification of connected Hopf algebras with finite $\mathrm{GK\text{-}dim}$ to that of Lie bialgebra structures on nilpotent finite-dimensional Lie algebras. Milen Yakimov showed in his talk a parallel argument for triangular Lie bialgebras. Various properties of connected Hopf algebras and several examples with small $\mathrm{GK\text{-}dim}$ found recently in the literature are explained through this theorem.

3.1.2 Nichols algebras over abelian groups

Iván Angiono reported on recent work with Andruskiewitsch and Heckenberger.

Conjecture 3. If (V, c) is of diagonal type and $\mathrm{GK\text{-}dim}\, \mathcal{B}(V) < \infty$, then its GRS is finite. (Thus, its connected components are as in [21]; the corresponding Nichols algebras are either finite-dimensional, positive parts of quantized enveloping simple Lie algebras or superalgebras, or an example spotted by Yamane).

Partial answers supporting this conjecture: when either $\dim V = 2$ or V is of affine Cartan type. Next, the blocks are braided vector spaces $\mathcal{V}(\epsilon, \ell)$, where $\epsilon \in \Bbbk^\times$ and $\ell \in \mathbb{N}_{\geq 2}$, the braiding being given in terms of a Jordan block of size ℓ with ϵ in the diagonal.

Theorem 9. $\operatorname{GK-dim} \mathcal{B}(\mathcal{V}(\epsilon, \ell)) < \infty$ *if and only if* $\ell = 2$ *and* $\epsilon^2 = 1$, *where* $\operatorname{GK-dim} \mathcal{B}(\mathcal{V}(\epsilon, \ell)) = 2$.

The algebra $\mathcal{B}(\mathcal{V}(1, 2))$ is a quadratic domain (the Jordan plane), while the presentation of $\mathcal{B}(\mathcal{V}(-1, 2))$ contains a cubic relation and the square of an element. The next class is those braided vector spaces with two indecomposable components, a block and a point. Those with finite GK-dim Nichols algebra are known.

Example 4. Let $\mathcal{G} \in \mathbb{N}$, $q \in \Bbbk^\times$. Let $\mathcal{L}(1, \mathcal{G})$ be the braided vector space with basis $(x_i)_{1\leq i\leq 3}$ and braiding

$$(c(x_i \otimes x_j))_{1\leq i,j\leq 3} = \begin{pmatrix} x_1 \otimes x_1 & (x_2 + x_1) \otimes x_1 & qx_3 \otimes x_1 \\ x_1 \otimes x_2 & (x_2 + x_1) \otimes x_2 & qx_3 \otimes x_2 \\ q^{-1}x_1 \otimes x_3 & q^{-1}(x_2 - \frac{\mathcal{G}}{2}x_1) \otimes x_3 & x_3 \otimes x_3 \end{pmatrix}. \tag{1}$$

Let $z_n = (ad_c x_2)^n x_3$. Then $\mathcal{B}(\mathcal{L}(1, \mathcal{G}))$ is a domain with $\operatorname{GK-dim} = \mathcal{G} + 3$, and

$$\mathcal{B}(\mathcal{L}(1, \mathcal{G})) \simeq \Bbbk\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,\ x_1x_3 - qx_3x_1,\ z_{\mathcal{G}+1},\ z_tz_{t+1} - q^{-1}\, z_{t+1}z_t,\ 0 \leq t < \mathcal{G}\rangle.$$

Actually, this is the only Nichols algebra of *a block plus a point* that is a domain with finite $\operatorname{GK-dim}$.

Assuming that the conjecture is true, it is possible to classify Nichols algebras with finite $\operatorname{GK-dim}$ over abelian groups, i.e. of braided vector spaces whose indecomposable components are either blocks or points.

3.1.3 Nichols algebras over non-abelian groups

The activity in this area seems to be restricted yet to finite-dimensional Nichols algebras over finite non-abelian groups. There are three directions of research: first, a few examples were explicitly computed in the last years (notably the so-called Fomin-Kirillov algebras $\mathcal{FK}_n$, that are finite-dimensional and quadratic for $3 \leq n \leq 5$, but unknown for $n \geq 6$). Second, several families of simple Yetter-Drinfeld modules over finite non-abelian groups have infinite-dimensional Nichols algebras (Andruskiewitsch, Carnovale, Fantino, García, Graña, Vendramin). Third, it is conjectured that the examples found so far exhaust all possibilities (Heckenberger, Lochmann, Vendramin). Leandro Vendramin reported on recent work with Heckenberger:

Theorem 10. [22, 23] *Under some mild technical assumptions, the classification of the finite-dimensional Nichols algebras of completely reducible Yetter-Drinfeld modules over finite non-abelian groups is known.*

The proof makes use of the delicate combinatorial notions of GRS and Weyl groupoid (Heckenberger–Yamane), already used in the abelian case, and the classification of the finite ones (Cuntz–Heckenberger). Hans-Jürgen Schneider gave a thorough explanation in his talk of the *raison-d'etre* of the Weyl groupoid.

Question 18. What are all infinite-dimensional Nichols algebras over finite groups with finite GK-dim?

Actually, it seems that the only known examples are of diagonal type.

3.2 Quantum symmetries

3.2.1 (Non-)existence of quantum actions

Chelsea Walton reported on this basic yet intriguingly difficult question. In this section, H is a finite-dimensional Hopf algebra acting inner faithfully on an algebra A. For simplicity, we assume that $\operatorname{char} \Bbbk = 0$. By Theorem 6, a meaningful initial step is to consider actions on division rings. Now, H is *Galois-theoretical* if it acts inner faithfully on a field [18]. This property is preserved under taking a Hopf subalgebra, and under tensor product, but not under Hopf dual, 2-cocycle deformations, nor Drinfeld twists.

Theorem 11.
- [17] *A semisimple Galois-theoretical Hopf algebra is necessarily a group algebra.*

- [18] *Taft algebras, $u_q(sl_2)$, some Drinfeld twists of small quantum groups, and the dual of a specific generalized Taft algebra (which is neither pointed nor semisimple) are Galois-theoretical.*
- [18] *Generalized Taft algebras, some book Hopf algebras, $\operatorname{gr} u_q(sl_2)$ are **not** Galois-theoretical.*

Here are some further results:

- [7, 8] If A is a Weyl algebra, then H is a group algebra.
- [6] The algebra of functions $H = \Bbbk^G$ on a finite group G acts inner faithfully on a central division algebra of degree d containing $\Bbbk$ iff G contains a normal abelian subgroup of index dividing d.

3.2.2 Quantum invariant theory

In this section, H is a finite-dimensional Hopf algebra and A is a graded connected AS-regular H-module algebra; in particular the action of H preserves the grading. The action is assumed inner-faithful. Let $n = \operatorname{GK-dim} A$. Ellen Kirkman reported the following advances in this line; a reference is the survey [24].

◊ When is A^H AS-regular? (In such case, we say that H is a *reflection* Hopf algebra).

To start with, suppose $H = \Bbbk G$, where G is a finite group of graded automorphisms and $g \in G$. Set $\operatorname{Tr}_A(g,t) = \sum_{j=0}^{\infty} \operatorname{tr}(g_{|A_j})t^j \in \Bbbk[[t]]$. Then g is a *reflection* if $\operatorname{Tr}_A(g,t) = \dfrac{1}{(1-t)^{n-1}q(t)}$, with $q(t) \neq 1$ [25]. If A is generated by V (the component of degree 1), then g is said to be a classical reflection if its restriction to V is a linear reflection; also, g is a mystic reflection if its restriction to V has invariant subspace of dimension $n-2$, and also eigenvalues $\pm\sqrt{-1}$.

Theorem 12. [25] *If A is a quantum polynomial ring, then any reflection is either classical or mystic.*

Motivated by the Chevalley-Sheppard-Todd theorem, the authors of [25] state:

Conjecture 4. $H = \Bbbk G$ is a reflection Hopf algebra if and only if G is generated by reflections.

The conjecture is supported by various partial answers. Other examples of reflection Hopf algebras (for suitable A), are e.g. the 8-dimensional Kac-Paliutkin Hopf algebra and the Sweedler Hopf algebra.

Question 19. Can reflection Hopf algebras be characterized?

◊ When is A^H AS-Gorenstein?

There is again a homological determinant $\operatorname{hdet}_H A : H \to \Bbbk$, defined in terms of local cohomology.

Theorem 13. [26] *If H is semisimple and the homological determinant is trivial, then A^H is AS-Gorenstein.*

There are partial converses, and a characterization of the AS-Gorensteinness of A^H by its Hilbert series.

◊ When is A^H complete intersection?

It is not yet clear what is the right version of *complete intersection* in the non-commutative setting; several different definitions that are equivalent in the commutative context cease to be so in general. Nevertheless some consequences on A^H can be shown in examples under some of those conditions.

3.3 Fusion categories

3.3.1 Verlinde categories

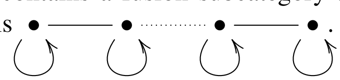
Let $\Bbbk$ be a field, $\operatorname{char} \Bbbk = p > 0$. Victor Ostrik reported on the Verlinde category $\operatorname{Ver}_p = \underline{\operatorname{Rep}} \Bbbk\mathbb{Z}/p$, see §2.4. (Notice that $\Bbbk\mathbb{Z}/p$ is a Nichols algebra). Here $\operatorname{Indec} \Bbbk\mathbb{Z}/p = \{1, \dots, p\}$ hence $\operatorname{Irr} \operatorname{Ver}_p = \{1, \dots, p-1\}$. For instance $\operatorname{Ver}_2 = \operatorname{Vec}_{\Bbbk}$, while $\operatorname{Ver}_3 = \operatorname{superVec}_{\Bbbk}$ (the category of super vector spaces).

Question 20. Identify Ver_p, for $p = 5, 7, \dots$.

The Verlinde category Ver_p is a symmetric fusion category. The graph of the multiplication by the indecomposable of dimension 2 in the Grothendieck ring is $\bullet$ —— $\bullet$ ⋯⋯ $\bullet$ —— $\bullet$, $p-1$ vertices.

Theorem 14. *Every symmetric fusion category $\mathcal{C}$ admits a unique symmetric tensor functor $\mathcal{C} \to \mathrm{Ver}_p$.*

Question 21. What are the semisimple (affine or finite) group schemes in Ver_p?

Example 5. (Etingof). Let $\mathfrak{g}$ be a two-dimensional Lie superalgebra, generated by y (even) and x (odd) with bracket $[x, x] = y$. Then $\underline{\mathrm{Rep}}\,\mathfrak{g}$ contains a fusion subcategory $\mathcal{E}$ with a generator whose graph of multiplication in the Grothendieck ring is  .

Here is a generalization of the Verlinde categories.

Example 6. (John Wood). Let $\mathcal{C} = \mathrm{Rep}\,\Bbbk\mathbb{Z}/p^n$. Then $\mathrm{Indec}\,\mathcal{C} \simeq \{1, 2, \dots, p^n\}$ (the parametrization is given by the dimension of a Jordan cell). Hence $\underline{\mathcal{C}} = \underline{\mathrm{Rep}}\,\Bbbk\mathbb{Z}/p^n$ has $p^n - p^{n-1}$ irreducibles (excluding those with $\dim = 0$).

Theorem 15. *(John Wood). $\underline{\mathcal{C}} \simeq \mathrm{Ver}_p \boxtimes \mathcal{E}' \boxtimes \mathcal{E}'' \boxtimes \cdots \boxtimes \mathcal{E}^{(n-1)}$, where the Grothendieck rings of the categories $\mathcal{E}^{(j)}$ are all isomorphic: $K(\mathcal{E}') \simeq K(\mathcal{E}'') \simeq \cdots \simeq K(\mathcal{E}^{(n-1)}) \simeq K(\mathcal{E})$.*

3.3.2 Group-theoretical categories

Here are some questions proposed by Ostrik.

Question 22. Let G be a finite group and V be an indecomposable representation of G over $\Bbbk$. What is the subcategory of $\underline{\mathrm{Rep}}\,\Bbbk G$ generated by G?

Question 23. Let $\Bbbk$ and G be as above. When is $\underline{\mathrm{Rep}}\,\Bbbk G$ a fusion category? Equivalently, when are there only finitely indecomposable G-modules with non-zero quantum dimension?

By a classical result of Higman, the group algebra $\Bbbk G$ has finite representation type if and only if the p-Sylow subgroups of G are cyclic.

Question 24. Same as Question 23 but for $\mathcal{C}(G, F, 1, 1)$, where $F < G$.

Note that $\mathcal{C}(G, F, 1, 1)$ has finite representation type if and only if the p-Sylow subgroups of F are cyclic.

3.3.3 The Brauer-Picard group

The Brauer-Picard group $\mathsf{BrPic}\mathcal{C}$ of a finite tensor category $\mathcal{C}$ is the group of (iso-classes of) invertible $\mathcal{C}$-bimodule categories [14]. If $\mathcal{C}$ is actually braided, then there is a subgroup $\mathsf{Pic}\mathcal{C}$ of $\mathsf{BrPic}\mathcal{C}$ consisting of classes of invertible left $\mathcal{C}$-module categories. If $Z(\mathcal{C})$ denotes the Drinfeld center of $\mathcal{C}$, then $\mathsf{BrPic}\mathcal{C} \simeq \mathsf{Pic}Z(\mathcal{C})$. A more flexible description is $\mathsf{BrPic}\mathcal{C} \simeq \mathsf{Aut}^{br}(Z(\mathcal{C}))$ [10, 14]; the last being a generalization of the classical orthogonal group, it brings geometric insights to the question. Here are some examples:

- [33] Let G be a finite group without normal abelian subgroups (e.g., a simple non-abelian group). Then $\mathsf{BrPic}(\mathrm{Vec}_G) \simeq H^2(G, k^\times) \rtimes \mathrm{Out}(G)$.
- [1] $\mathsf{BrPic}(\mathrm{Rep}\,\Lambda(k^n) \rtimes k\mathbb{Z}/2\mathbb{Z}) \simeq PSp_{2n}(k) \times Z/2\mathbb{Z}$.

3.3.4 The Frobenius-Schur exponent

Siu-Hung Ng gave an overview of this notion. The classical definition of Frobenius-Schur indicator of a finite group was generalized to Hopf and quasi-Hopf algebras (in various ways), and ultimately in the context of spherical fusion categories (Ng–Schauenburg). Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$, $V \in \mathcal{C}$, $n \in \mathbb{N}$. The n-th Frobenius-Schur indicator $\nu_n(V)$ is defined in terms of the trace of the spherical structure, as well as the dimension $\dim V$ of V. There exists $N \in \mathbb{N}$ such that $\nu_N(V) = \dim V$ for all $V \in \mathcal{C}$ (Ng–Schauenburg). The least such N is called the *Frobenius-Schur exponent* of $\mathcal{C}$; notice that it may differ from the exponent of $\mathcal{C}$ defined by Etingof, but not widely. Here is a version of the Cauchy Theorem:

Theorem 16. *[5] The principal ideals generated by* $(\dim \mathcal{C})^2$ *and* N *have the same prime factors in* $\mathbb{Z}[e^{\frac{2\pi i}{N}}]$.

This result is a crucial step in the following remarkable rigidity theorem.

Theorem 17. *[5] There are only finitely many modular categories of any fixed rank, up to equivalence.*

3.3.5 Spherical and modular fusion categories

Eric Rowell chaired a Round Table on modular fusion categories and proposed the following problems.

Question 25. Are there only finitely many modular categories of any fixed rank, up to equivalence?

Conjecture 5. If $\mathcal{C}$ is a braided fusion category and $X \in \mathcal{C}$, then the image of the braid group $\mathbb{B}_n$ in $\operatorname{End} X^{\otimes n}$ is finite for all $n \geq 2$, if and only if the Perron-Frobenius dimension of X satisfies $(\text{FP-}\dim X)^2 \in \mathbb{Z}$.

This stems from a paper by Etingof, Rowell and Witherspoon, with partial answers. Another question: If (V, c) is a unitary braided vector space with c of finite order, is the image of $\mathbb{B}_n$ in $\operatorname{End} V^{\otimes n}$ finite?

3.3.6 Jordan-Hölder theorems

Sonia Natale reported two Jordan-Hölder theorems, for Hopf algebras and for weakly group-theoretical categories [31, 32]. In both cases, any object has uniquely defined composition factors and length. The composition factors are simple Hopf algebras in the first case, and simple groups in the second. But, if H is a weakly group-theoretical Hopf algebra, then the composition factors of H and of $\operatorname{Rep} H$ do not coincide necessarily.

4 Scientific Progress Made and Outcome of the Meeting

The participants participated very actively in informal discussions and exchanges. Some very popular topics were cohomology of Hopf algebras (a Round Table was devoted to this) and the Question 17.

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