Mahler measure, regulators and modular units

Marie José BERTIN

Institut Mathématique de Jussieu
Université Paris 6
4 Place Jussieu 75005 PARIS

marie-jose.bertin@imj-prg.fr

Banff, October 5, 2015

We are interested in Boyd's conjectures relating the logarithmic Mahler measure of certain two-variables polynomials defining elliptic curves and the L-series at $s = 2$ of this elliptic curve.

Introduction

The logarithmic Mahler measure m of a non-zero Laurent polynomial $A \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ is defined as

$$m(A) := \int_0^1 \dots \int_0^1 \log \mid A(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}) \mid d\theta_1 ... d\theta_n$$

and its Mahler measure is the exponential of the latter.
If $A(x, y)$ is in two variables we can write

$$A(x, y) = a_0(y) \prod_{j=1}^{d} (x - x_j(y))$$

with $x_j(y)$ algebraic functions in y.

By Jensen's formula

$$m(A) = m(a_0) + \sum_{j=1}^{d} \frac{1}{2\pi i} \int_{|y|=1} \log^+ \mid x_j(y) \mid \frac{dy}{y}$$

where $\log^+ \mid z \mid = \log \mid z \mid$ if $\mid z \mid \geq 1$ and 0 otherwise.
Defining

$$\eta(x, y) := \log \mid x \mid d \arg y - \log \mid y \mid d \arg x$$

a real differential 1-form on $X \setminus S$ (X the variety defined by the polynomial A, smooth projective completion of Y zero locus of A, S points of X where x or y has a zero or a pole), we get

$$m(A) = m(a_0) + \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

γ oriented path on X projecting to $Y \cap \{\mid y \mid = 1, \mid x \mid \geq 1\}$

Deninger's guess (1996)

To illustrate which kind of polynomials and which kind of conjectures or results, let us give Deninger's guess (1996) proved in 2011 by Rogers and Zudilin, then again by Zudilin in 2013

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) = \frac{15}{4\pi^2} L(E, 2) =: L'(E, 0) = b_{15}$$

The elliptic curve E is $15a8$ (Cremona's notation) of conductor 15 defined by

$$Y^2 + XY + Y = X^3 + X^2$$

Its L-series is given by the modular form

$$f_{15A}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

The polynomial

$$x + \frac{1}{x} + y + \frac{1}{y} + 1$$

is tempered

"Tempered" means the roots of all the face polynomials of the Newton polygon of A are roots of unity.

The polynomial

$$Y^2 + XY + Y - (X^3 + X^2)$$

is also tempered.

Very important to obtain formulas "à la Deninger".

Just after Deninger's guess, Boyd obtained a lot of conjectures based on numerical computations.

What is proved now concerning non CM elliptic curves?

Denote

$$b_{N_k} := \frac{N_k}{4\pi^2} L(E^k, 2) \quad m_k = m(P_k)$$

with N_k conductor of the elliptic curve E^k defined by P_k.

In the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + k$$

Zudilin-Rogers (2011) then Zudilin (2013) $m(1) = b_{15}$

Zudilin arxiv (2013) $m_{2i} = b_{40}$, $m_2 = b_{24}$, $m_i = 2b_{17}$, $m_{\sqrt{2}} = \frac{1}{4} b_{56}$

Brunault arxiv (april 2015) $m_3 = 2b_{21}$, $m_{12} = 2b_{48}$

Lalin-Samart-Zudilin arxiv (july 2015) $m_3 = 2b_{21}$ (another proof)

In the family

$$P_k = x^3 + y^3 + 1 - kxy$$

Mellit preprint (2009) arxiv (2012) $m_{-1} = 2b_{14}$, $m_5 = 7b_{14}$

In the family

$$P_k = (x+1)y^2 + (x^2 + kx + 1)y + x(x+1)$$

Mellit preprint (2009) arxiv (2012) $m_1 = b_{14}$, $m_{-5} = 6b_{14}$, $m_{10} = 10b_{14}$

In the family

$$P_k = y^2 + kxy + y - x^3$$

Brunault arxiv (april 2015) $m_{-1} = 2b_{14}$, $m_{-2} = b_{35}$, $m_{-3} = b_{54}$

Related also to the family

$$P_k(x,y) = (x+1)y^2 + (x^2+kx+1)y + x(x+1)$$

Boyd conjectured the two formulae

$$m_4 = 3b_{20} \quad \text{and} \quad m_{-2} = 2b_{20}.$$

In fact, E^4 is isomorphic to the curve $20a2$ $[0,1,0,-1,0]$, E^{-2} is isomorphic to the curve $20a1$ $[0,1,0,4,4]$, 2-isogenous to $20a2$. The corresponding modular form on $\Gamma_0(20)$ thus giving the L-series is

$$f_{20A} = \eta(2z)^2\eta(10z)^2 = q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + 2q^{15} \ldots$$

We proved recently these conjectures and the main ingredients are regulators and modular units

“tempered” and the K_2 of the elliptic curve

Let X be a smooth projective algebraic curve defined over $\mathbb{C}$ and let $\mathbb{C}(X)$ be its function field. Let x, $y \in \mathbb{C}(X)$ be two non-constant rational functions and let $S \subset X$ be the set of zeros and poles of x or y. The image of the rational map $(x, y) : X \setminus S \to \mathbb{C}^* \times \mathbb{C}^*$ is of dimension 1; let $A \in \mathbb{C}[x, y]$ be a defining equation.

$$\{x, y\} \in K_2(X) \otimes \mathbb{Q} \Leftrightarrow A \text{“tempered”}$$

Integral expression of the regulator

The regulator r can be expressed as an integral

$$\begin{array}{rccc} r: & K_2(E) & \to & \mathbb{C} \\ & \{f, g\} & \mapsto & \frac{1}{2\pi}\int_\gamma \eta(f,g) \end{array}$$

with

$$\eta(f,g) = \log|f| d(\arg g) - \log|g| d(\arg f),$$

f and $g \in \mathbb{Q}(E)$ and γ closed path not going through zeros and poles of f and g and generating the subgroup of cycles $H_1(E, \mathbb{Z})^-$

Regulator and Mahler measure

The Mahler measure can be expressed as a regulator if we can prove that the path of integration in the expression of the Mahler measure belongs to $H_1(E, \mathbb{Z})^-$.

This is precisely the case for the polynomial P_{-2}.

Set $P_{-2}(x_2, y_2)$ the polynomial

$$P_{-2}(x_2, y_2) = (x_2 + 1)y_2^2 + (x_2^2 - 2x_2 + 1)y_2 + x_2(x_2 + 1).$$

Then

$$2m_{-2} = \pm r(\{x_2, y_2\}).$$

The diamond operator

Let $\mathbb{Z}[\langle P\rangle]$ the subgroup of $\mathbb{Z}[E(\mathbb{Q})]$ generated by $P \in E(\mathbb{Q})$ and $\mathbb{Z}[E(\mathbb{Q})]^-$ its quotient by the relation $cl(-P) = -cl(P)$.
Define

$$\begin{array}{rccc} \diamond : & \mathbb{Z}[\langle P\rangle] \times \mathbb{Z}[\langle P\rangle] & \to & \mathbb{Z}[E(\mathbb{Q})]^- \\ & ((f),(g)) & \mapsto & (f)\diamond(g) = \sum_{m,n} a_n b_m cl((n-m)P) \end{array}$$

$(f) = \sum_{n\in\mathbb{Z}} a_n[nP]$,$(g) = \sum_{n\in\mathbb{Z}} b_n[nP]$

The elliptic dilogarithm (introduced by Bloch)

E elliptic curve on $\mathbb{Q}$
On $E(\mathbb{C})$, we have the representations

$$\begin{array}{ccccc} E(\mathbb{C}) & \stackrel{\sim}{\to} & \mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}) & \stackrel{\sim}{\to} & \mathbb{C}^*/q^{\mathbb{Z}} \\ (\mathfrak{P}(u),\mathfrak{P}'(u)) & \mapsto & u(mod \Lambda) & \mapsto & z=\exp 2\pi i u \end{array}$$

The elliptic dilogarithm D^E is

$$D^E(P)=\sum_{n=-\infty}^{+\infty} D(q^n z)$$

where D denotes the Bloch-Wigner dilogarithm.

Bloch's Theorem: regulator and elliptic dilogarithm

Theorem

Let f and g functions on the elliptic curve E with divisors elements of $\mathbb{Z}[\langle P\rangle]$ such that $\{f,g\} \in K_2(E)$, then

$$\pi r(\{f,g\}) = D^E((f) \diamond (g))$$

Touafek's results

Some years later (2008), in his thesis (not published in extenso), Touafek considered the elliptic curve E_2 defined by the equation

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3$$

exhibited the isomorphisms between E_2, 20a1 and E^{-2}, remarked that

$$\{X_2, Y_2\} \in K_2(E_2) \otimes \mathbb{Q}$$

$$\{x_2, y_2\} \in K_2(E^2) \otimes \mathbb{Q}$$

and used Bloch's theorem to derive the equality

$$r(\{X_2, Y_2\}) = r(\{x_2, y_2\})$$

and conjectured their common value $4b_{20}$.

Beilinson's result, Zagier's conjecture

For elliptic modular curves E, Beilinson proved

$$L(E,2) = \pi D^E(\xi), \quad \xi \in \mathbb{Z}[E(\mathbb{C})]_{\text{tors}}$$

For a general elliptic curve E, Zagier conjectured

$$L(E/\mathbb{Q},2) \stackrel{?}{=} \pi D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})]^{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}$$

Proof of Touafek's conjecture: modular units

The idea is the parametrization by modular units. (Brunault, Mellit, Zudilin).

Recall that a modular unit is a modular function whose all zeros and poles are cusps, for example certain quotient of eta functions for $\Gamma_0(20)$.

We proved the lemma

Lemma

The elliptic curve E_2 defined by

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3$$

is isomorphic to the curve '20a1' $[0, 1, 0, 4, 4]$ *in Cremona's classification and can be parametrized by eta quotients, modular units on $X_0(20)$. More precisely*

$$\begin{aligned} X_2 &= \frac{\eta(4\tau)^4}{\eta(20\tau)^4}\frac{\eta(10\tau)^2}{\eta(2\tau)^2} \\ Y_2 &= -\frac{\eta(4\tau)}{\eta(\tau)}\frac{\eta(5\tau)^5}{\eta(20\tau)^5} \end{aligned}$$

Modular units

Let us recall first the definition of the modular unit g_a:

$$g_a(\tau) := q^{NB(a/N)/2} \prod_{\substack{n \geq 1 \\ n \equiv a \bmod N}} (1 - q^n) \prod_{\substack{n \geq 1 \\ n \equiv -a \bmod N}} (1 - q^n)$$

Now it follows from the definition of a modular unit:

$$X_2 = \left(\frac{g_4 g_8}{g_2 g_6}\right)^2$$

$$Y_2 = -\frac{g_5^4 g_{10}^2}{g_1 g_2 g_3 g_6 g_7 g_9}$$

Main ingredient: proof by Zudilin

Theorem

For integers a, b, c with ac and bc not divisible by N, we have the formula

$$\int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi} L(f(\tau) - f(i\infty), 2)$$

where $f(\tau) = f_{a,b;c}(\tau)$, $f_{a,b;c} := e_{a,bc}e_{b,-ac} - e_{a,-bc}e_{b,ac}$ *and*

$$e_{a,b}(\tau) = \frac{1}{2}\left(\frac{1+\zeta_N^a}{1-\zeta_N^a} + \frac{1+\zeta_N^b}{1-\zeta_N^b}\right) + \sum_{m,\, n\geq 1} (\zeta_N^{am+bn} - \zeta_N^{-(am+bn)})q^{mn}$$

$\zeta_N := \exp(2\pi i/N)$, $q := \exp(2\pi i\tau)$.

How to choose c: the path of integration

If $\alpha,\ \beta \in \mathcal{H}^*$ satisfy $\beta = M(\alpha)$, $M \in \Gamma_0(N)$ (α and β are said equivalent under the action of $\Gamma_0(N)$).
Any smooth path (for instance a geodesic path) projects to a closed path in the quotient space $X_0(N)$, hence determines an integral homology class in $H_1(X_0(N), \mathbb{Z})$, which depends only on α and β and not on the path chosen. In fact the class depends only on the matrix M. This homology class is denoted by the modular symbol $\{\alpha, \beta\}_{\Gamma_0(N)}$. Conversely, every homology class $\gamma \in H_1(X_0(N), \mathbb{Z})$ can be represented by such a modular symbol $\{\alpha, \beta\}_{\Gamma_0(N)}$.
For $f \in S_2(\Gamma_0(N))$,

$$< \gamma, f >:= \int_\gamma 2\pi i f(z)dz = 2\pi i \int_\alpha^\beta f(z)dz$$

is called a period of the cusp form f.
Elements of $H_1^-(X_0(N), \mathbb{R})$ are identified by

$$< \gamma, f > \in i\mathbb{R} \iff \gamma \in H_1^-(X_0(N), \mathbb{R}).$$

Recall also that by the Manin-Drinfeld theorem, the rational homology $H_1(X_0(N),\mathbb{Q})$ is generated by paths between cusps.
The closed path of integration γ generating $H_1(E,\mathbb{Z})^-$ in the expression of the regulator becomes under the parametrization a closed path generator of $H_1^-(X_0(20),\mathbb{Z})$, hence an appropriate modular symbol we can compute using Sage. We can take the closed path $\{-3/20, 3/20\}$ and apply theorem B-M-Z. So

$$
\begin{aligned}
&r(\{X_2, Y_2\})\\
&= \frac{1}{2\pi}\left(\int_{-3/20}^{i\infty} - \int_{3/20}^{i\infty}\right)\eta(\left(\frac{g_4 g_8}{g_2 g_6}\right)^2, \frac{g_5^4 g_{10}^2}{g_1 g_2 g_3 g_6 g_7 g_9})\\
&= \frac{1}{2\pi}\frac{4}{4\pi}(4L(f_{4,5;-3}) + 2L(f_{4,10;-3}) - L(f_{4,1;-3}) - L(f_{4,2;-3})\\
&-L(f_{4,3;-3}) - L(f_{4,6;-3}) - L(f_{4,7;-3}) - L(f_{4,7;-3})\\
&+4L(f_{8,5;-3}) + 2L(f_{8,10;-3}) - L(f_{8,1;-3}) - L(f_{8,2;-3}) - L(f_{8,3;-3})\\
&-L(f_{8,6;-3}) - L(f_{8,7;-3}) - L(f_{8,7;-3})\\
&...)\\
&= \frac{1}{4\pi^2}4 \times 20L(f,2)
\end{aligned}
$$

f is the newform of conductor 20

$$f(q) = q - 2q^3 - q^5 + 2q^7 + q^9 + \dots$$

The end

We have just proved Touafek's conjecture

$$r(\{X_2, Y_2\}) = \frac{1}{2\pi^2} 40 L(f, 2) = 4 b_{20}$$

and previously it was obtained

$$r(\{X_2, Y_2\}) = r(\{x_2, y_2\})$$

$$2 m_{-2} = \pm r(\{x_2, y_2\}).$$

We deduce Boyd's conjecture

$$m_{-2} = m(P_{-2}) = 2 b_{20}$$

where $b_{20} = \frac{20}{4\pi^2} L(E^{-2}, 2)$.

Proof of the second conjecture

Similarly, Touafek considered the isomorphic curves E^4 defined by

$$(x_1+1)y_1^2+(x_1^2+4x_1+1)y_1+x_1(x_1+1)=0$$

and the elliptic curve E_1 defined by

$$Y_1^2+2X_1Y_1-X_1^3+X_1=0.$$

Both polynomials are tempered; so the respective regulators $r(\{x_1,y_1\}$ and $r(\{X_1,Y_1\})$ can be defined and from Touafek's computations we can also deduce the equality

$$r(\{x_1,y_1\})=\frac{3}{2}r(\{X_1,Y_1\}).$$

Touafek proved also the relation

$$r(\{X_2,Y_2\})=r(\{X_1,Y_1\}).$$

As previously we get

$$2m_4=r(\{x_1,y_1\}).$$

Finally, it follows

$$
\begin{aligned}
2m_4 = r(\{x_1, y_1\}) &= \tfrac{3}{2} r(\{X_1, Y_1\}) \\
&= \tfrac{3}{2} r(\{X_2, Y_2\}) \\
&= \tfrac{3}{2} 4 b_{20}
\end{aligned}
$$