Sets of Pisot-like numbers

Let $r > 0$, and S_r be the set of real positive algebraic integers $\alpha > r$, all of whose conjugates $\neq \alpha$ are $< r$. Clearly S_1 is the set of Pisot numbers – known to form a closed subset of $\mathbb{R}$ (Salem 1944).

Lemma 1. If $r < 1$ then S_r is discrete (and so certainly closed).

Proof. Let n denote the degree of $\alpha \in S_r$, where $r < 1$. Then

$$1 \leq |\operatorname{Norm}(\alpha)| \leq \alpha r^{n-1},$$

so that $\alpha > \left(\frac{1}{r}\right)^{n-1}$. Hence the degree of such α in $(1, B]$ is at most $1 + \log B / \log\left(\frac{1}{r}\right)$. Since there are only a finite number of monic integer polynomials of degree at most n having all zeros in the disc $|z| \leq B$, we have that there are only finitely many α in $S_r \cap (1, B]$. Thus S_r is discrete.

Lemma 2. The union $\cup_{r>0} S_r$ is dense in $(1, \infty)$.

Proof. For positive integers n, b, let $p(z) = z^{2n+1}(z^{2n} - b) - 1$. Then p has a unique zero $\alpha > 1$, and all other zeros have smaller modulus than α. Thus α is a Perron number, and so it belongs to S_α.

Now take any $t > 1$, and put $b = \lfloor t^{2n} \rfloor$. Then as $n \to \infty$ for fixed t, the largest root of $p(z) = z^{2n+1}(z^{2n} - \lfloor t^{2n} \rfloor) - 1$ tends to t. Hence $\cup_{r>0} S_r$ is dense in $(1, \infty)$.

Question. For $r > 1$, is S_r dense in $(1, \infty)$? Is it closed? Or what?