On the degree of compositum of two number fields

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Sum-feasible triplets

Sum-feasible triplet $(a, b, c) \in \mathbb{N}^3$:

- α, β, γ – algebraic numbers,
- $\deg \alpha = a$, $\deg \beta = b$, $\deg \gamma = c$,
- $\alpha + \beta + \gamma = 0$.

For example, $(2, 2, 4)$ is sum-feasible:

$$\alpha = \sqrt{2}, \quad \beta = \sqrt{3}, \quad \gamma = -(\sqrt{2} + \sqrt{3}).$$

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Problem

[1] *Find all possible sum-feasible triplets* $(a, b, c) \in \mathbb{N}^3$.

Equivalent to:

Question

Given

- a, b – fixed positive integers,
- α, β – algebraic numbers, $\deg \alpha = a$, $\deg \beta = b$,

what are the possible values of $\deg(\alpha + \beta)$?

For example, if $\deg \alpha = \deg \beta = 5$ then

$$\deg(\alpha + \beta) \in \{1, 5, 10, 20, 25\}.$$

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Related results

Theorem (I. Kaplansky, 1969)

Suppose that

- α, β – *algebraic numbers,*
- $\deg\alpha > \deg\beta$,
- $\deg\alpha$ *is a prime number.*

Then $\deg(\alpha+\beta) = \deg\alpha\cdot\deg\beta$.

Theorem (I.M. Isaacs, 1970)

If a triplet (a, b, c) *is sum-feasible and gcd*$(a, b) = 1$ *then* $c = ab$.

As J. Browkin, B. Diviš and A. Schinzel remarked, the proof of Isaacs implies that if $[\mathbb{Q}(\alpha,\beta):\mathbb{Q}] = \deg\alpha\cdot\deg\beta$ then

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Compositum-feasible triplet $(a, b, c) \in \mathbb{N}^3$:

- K, L – number fields,
- $[K : \mathbb{Q}] = a$, $[L : \mathbb{Q}] = b$ and $[KL : \mathbb{Q}] = c$.

- For example, $(2, 2, 4)$ is compositum-feasible:

$$K = \mathbb{Q}(\sqrt{2}), \quad L = \mathbb{Q}(\sqrt{3}), \quad KL = \mathbb{Q}(\sqrt{2} + \sqrt{3}).$$

- compositum-feasible $\implies$ sum-feasible
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Theorem (P. D., A. Dubickas)

If the triplet $(a, b, c) \in \mathbb{N}^3$ is sum-feasible then it is also product-feasible.

- $(2, 3, 3)$ is product-feasible, but not sum-feasible
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Main results

Theorem (P. D., A. Dubickas, C. Smyth)

All the triplets (a, b, c) of positive integers with $a \leqslant b \leqslant c$, $b \leqslant 6$ that are sum-feasible are given in the following table.

b \ a	1	2	3	4	5	6
1	1					
2	2	2, 4				
3	3	6	3, 6, 9			
4	4	4, 8	12	4, 6, 8, 12, 16		
5	5	10	15	20	5, 10, 20, 25	
6	6	6, 12	6, 12, 18	6, 12, 24	30	6, 8?, 9, 12, 15, 18, 24, 30, 36

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- All these triplets are compositum-feasible, except for $(4,4,6)$, $(4,6,6)$, $(6,6,8)$, $(6,6,9)$ and $(6,6,15)$

Theorem (P. D., A. Dubickas, F. Luca[2])

- $(6,6,8)$ *is not sum-feasible*
- *The set of all the sum-feasible triplets* $(a,7,c)$, $a \leqslant 7 \leqslant c$:

$$\{(1,7,7),\ (2,7,14),\ (3,7,21),\ (4,7,28),\ (5,7,35),\ (6,7,42),\ (7,7,7),(7,7,14),(7,7,21),(7,7,28),(7,7,42),(7,7,49)\}$$

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Proposition (P. D., A. Dubickas, F. Luca)

A triplet $(a, b, c) \in \mathbb{N}^3$ *is compositum-feasible if and only if there exists an irreducible polynomial* $f(x) \in \mathbb{Q}[x]$ *of degree* c *such that the Galois group* G *of its splitting field has two subgroups* H_1 *and* H_2 *such that* $[G : H_1] = a$, $[G : H_2] = b$ *and* $[G : H_1 \cap H_2] = c$.

- In principle, one can decide whether a given (a, b, c) is compositum-feasible by performing a finite computation
- difficult to use in practice, unless c is very small
- $(7, 7, 28)$ is compositum-feasible; $\mathrm{PSL}(2, 7)$

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Exponent triangle inequality

Let p – prime number, n – a positive integer. Recall that the nonnegative integer $\mathrm{ord}_p(n)$ is defined by

$$p^{\mathrm{ord}_p(n)} \mid n \quad \text{and} \quad p^{\mathrm{ord}_p(n)+1} \nmid n.$$

We say that a triplet (a, b, c) satisfies the exponent triangle inequality with respect to a prime number p if

$$\mathrm{ord}_p(a) + \mathrm{ord}_p(b) \geqslant \mathrm{ord}_p(c), \quad \mathrm{ord}_p(b) + \mathrm{ord}_p(c) \geqslant \mathrm{ord}_p(a) \quad \text{and}$$

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Theorem (P. D., A. Dubickas, C. Smyth)

If a triplet of positive integers (a, b, c) satisfies the exponent triangle inequality with respect to every prime number then the triplet (a, b, c) is sum-feasible.

- the condition is not necessary: $(3, 3, 6)$ is sum-feasible
- $(a, b, c) := \left(2^m + 1, 2^m + 1, 2^m(2^m + 1)\right)$ is sum-feasible and

$$\mathrm{ord}_2(c) - \mathrm{ord}_2(b) - \mathrm{ord}_2(a) = m$$

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Suppose that positive integers a, b and c satisfy $a \mid c$, $b \mid c$ and $c \mid ab$. Then the triplet (a, b, c) is compositum-feasible.

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$$\left(a(abc)^m,\ b(abc)^m,\ c(abc)^{m+1}\right)$$

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For every integer $\ell \geqslant 2$ *and every prime number* $p > \ell^2 - \ell + 1$ *the triplet* $(p, p, p(p - \ell))$ *is not product-feasible.*

$(7, 7, 28)$ corresponds to the values $p = 7$ and $\ell = 3$ for which $p = \ell^2 - \ell + 1$.

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Conjecture (P. D., A. Dubickas, C. Smyth)

If two triplets (a, b, c) and (a', b', c') are compositum-feasible then so is the triplet (aa', bb', cc').

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Theorem (P. D., A. Dubickas)

If every finite group occurs over $\mathbb{Q}$ *as a Galois group then the Conjecture is true.*

$(7, 7, 28)$ is compositum-feasible

Consider $p(x) = x^7 - 7x + 3$

L splitting field of $p(x)$ over $\mathbb{Q}$

Then $G :=\text{Gal}(L/\mathbb{Q})$ is isomorphic to PSL$(2, 7)$ – the second nonabelian simple group of order 168.

There exist subgroups H_1 and H_2 of G such that

$$[G : H_1] = [G : H_2] = 7 \text{ and } [G : H_1 \cap H_2] = 28.$$

Then the fixed fields L^{H_1}, L^{H_2} and $L^{H_1 \cap H_2}$ have degrees 7, 7, 28, respectively. Since $L^{H_1 \cap H_2} = L^{H_1} L^{H_2}$, the triplet $(7, 7, 28)$ is compositum-feasible.

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Thank you!