The Mahler measure of elliptic curves

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$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} \mathrm{m}(P) &= \int_0^1 \dots \int_0^1 \log \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right) \right| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

By Jensen's formula,

$$\mathrm{m}\left(a \prod (x - k_i)\right) = \log |a| + \sum \log \max\{1, |k_i|\}.$$

Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Volumes in hyperbolic space
- Entropy of certain arithmetic dynamical systems
- Special values of L-functions

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The measures of a family of genus-one curves

$$m(k) := \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Numerical Guess (Boyd (1998))

$$m(k) \stackrel{?}{=} \frac{b_{N(k)}}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

$$b_{N(k)} = \frac{N(k)}{4\pi^2} L(E_k, 2) = \varepsilon L'(E_k, 0)$$

$s_k \in \mathbb{Z}$ *(often)*

$$X = -\frac{1}{xy}, \qquad Y = \frac{(y-x)(1+xy)}{2x^2y^2}$$

$$E_{N(k)} : Y^2 = X^3 + (k^2/4 - 2)X^2 + X$$

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Boyd's conjectures

$$m(k) \stackrel{?}{=} \frac{N(k)}{4\pi^2 s_k} L(E_k, 2)$$

k	s_k	$N(k)$	k	s_k	$N(k)$
1	1	15	11	-8	1155
2	1	24	12	1/2	48
3	1/2	21	13	-4	663
4	*	*	14	8	840
5	1/6	15	15	-24	3135
6	2	120	16	1/11	15
7	2	231	17	-24	4641
8	1/4	24	18	-16	1848
9	2	195	19	-40	6555
10	-8	840	20	2	240

Some proven formulas for $m(k)$

$m\left(4\sqrt{2}\right) = b_{64}$	Rodriguez-Villegas (1997)
$m(1) = b_{15}$	Rogers & Zudilin (2010)
$m(2i) = b_{40}$	Mellit (2011)
$m(2) = b_{24}$	Rogers & Zudilin (2012)
$m(i) = 2b_{17}$	Zudilin (2014)
$m(\sqrt{2}) = \frac{1}{4}b_{56}$	Zudilin (2014)
$m(3) = 2b_{21}$	Brunault (2015)
$m(12) = 2b_{48}$	Brunault (2015)

L. & Rogers (2007), L. (2010)

$$m(5) = 6m(1), \quad m(16) = 11m(1), \quad m(3i) = 5m(1),$$

$$m(8) = 4m(2), \qquad m(3\sqrt{2}) = \frac{5}{2}m(2).$$

The relationship with Beilinson's conjectures

Deninger (1997): Understand these formulas in terms of Beĭlinson's conjectures.

Global information from local information through L-functions

$$L'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

$$\xi \in K \text{ rank-one abelian group}$$

Example: Dirichlet class number formula, for F real quadratic field,

$$\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\xi| \qquad \xi \in \mathcal{O}_F^*$$

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The relationship with the regulator

$$xy\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right)$$

$$P(x,y)=a_2(x)(y-y_1(x))(y-y_2(x)) \qquad E: P(x,y)=0 \quad \text{elliptic curve}$$

$$\mathrm{m}(P)-\mathrm{m}(a_2(x))=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log|y-y_1(x)|+\log|y-y_2(x)|\frac{dx}{x}\frac{dy}{y}$$

The relationship with the regulator

$$\mathrm{m}(P) - \mathrm{m}(a_2(x)) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log|y - y_1(x)| + \log|y - y_2(x)| \frac{dx}{x}\frac{dy}{y}$$

Suppose that $|y_1(x)| \geq 1$ and $|y_2(x)| \leq 1$.

By Jensen's formula,

$$= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |y_1(x)| \frac{dx}{x} = \frac{1}{2\pi i} \int_\gamma \log|y| \frac{dx}{x}$$

$$\gamma = \{(x, y) \in E, |x| = 1, |y| \geq 1\}$$

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The regulator

$$\begin{aligned} \mathrm{m}(P) - \mathrm{m}(a_2(x)) &= \frac{1}{2\pi i}\int_\gamma \log|y|\frac{dx}{x} \\ &= \frac{1}{2\pi}\int_\gamma \eta(x,y) \end{aligned}$$

$$\eta(x,y) = \log|x| d\arg y - \log|y| d\arg x$$

multiplicative and antisymmetric

Under favorable circumstances, $\int_\gamma \eta(x,y)$ is a function on

$$K_2(E) \times H_1(E,\mathbb{Z}).$$

The favorable cases

- Tame symbols are trivial ($\int_\gamma \eta(x,y)$ is a function on $K_2(E) \times H_1(E,\mathbb{Z})$).
- $E : P(x,y) = 0$ admits a parametrization by modular units $x(\tau)$ and $y(\tau)$, products and quotients of Siegel functions:

$$g_a(\tau) = q^{NB_2(a/N)/2} \prod_{\substack{n\geq 1 \\ n\equiv a \bmod N}} (1-q^n) \prod_{\substack{n\geq 1 \\ n\equiv -a \bmod N}} (1-q^n)$$

$$q = e^{2\pi i\tau} \qquad B_2(x) = \{x\}^2 - \{x\} + 1/6$$

Integrating η

Theorem (Mellit–Brunault–Zudilin (2014))

For $a, b, c \in \mathbb{Z}$, $N \nmid ac, bc$,

$$\int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi} L(f(\tau) - f(i\infty), 2)$$

where $f(\tau) = f_{a,b;c}(\tau)$ is a weight 2 modular form

$$f_{a,b;c} := e_{a,bc} e_{b,-ac} - e_{a,-bc} e_{b,ac}$$

where $e_{a,b}$ is a weight 1 level N^2 Eisenstein series.

The case $m(3)$

Theorem (Brunault, L-Samart-Zudilin, 2015)

$$m(3) = 2L'(E,0)$$

$$x + \frac{1}{x} + y + \frac{1}{y} + 3 = 0$$

does not have a modular unit parametrization.

Strategies

- Modify the curve via an isogeny to get another curve that can be parametrized by modular units, use this to prove $m(3)$.
- Brunault extends MBZ for the integral of $\eta(g_a, g_b)$ to Siegel units and works with Siegel units.

Extend the family

$$P_{a,c}(x) = a\left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + c,$$

$$X = -\frac{a}{xy}, \qquad Y = \frac{a}{2xy}\left(y - \frac{1}{y} - a\left(x - \frac{1}{x}\right)\right),$$

$$Y^2 = X\left(X^2 + \left(\frac{c^2}{4} - 1 - a^2\right)X + a^2\right).$$

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The 2-isogeny

$$a\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)+c=0 \qquad c=\frac{\sqrt{2}(a^2-1)}{\sqrt{a^2+1}}$$

There is a 2 isogeny φ between the latter and

$$x'+\frac{1}{x'}+y'+\frac{1}{y'}+\frac{4(a^2-1)}{a^2+1}=0.$$

Take $a=\sqrt{7}$, $c=3$,

$$E_{\sqrt{7},3} \text{ is isogenous to } E_{1,3}.$$

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A modular parametrization

$$\sqrt{7}\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)+3=0,$$

Furthermore:

$$\begin{aligned}\tilde{x}(\tau)&=\frac{1}{\sqrt{7}}\frac{\eta(\tau)\eta(3\tau)}{\eta(7\tau)\eta(21\tau)}\\&=\frac{1}{\sqrt{7}}g_1(\tau)g_2(\tau)g_3^2(\tau)g_4(\tau)g_5(\tau)g_6^2(\tau)g_8(\tau)g_9^2(\tau)g_{10}(\tau),\\\tilde{y}(\tau)&=-\left(\frac{\eta(\tau)\eta(21\tau)}{\eta(3\tau)\eta(7\tau)}\right)^2\\&=-(g_1(\tau)g_2(\tau)g_4(\tau)g_5(\tau)g_8(\tau)g_{10}(\tau))^2,\end{aligned}$$

$$\tau\in X_0(21).$$

More difficulties for the case $m(3)$

$$P_{\sqrt{7},3}(x,y) = \sqrt{7}\left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + 3.$$

$$y_\pm(x) = \frac{-(\sqrt{7}(x+x^{-1})+3)) \pm \sqrt{(\sqrt{7}(x+x^{-1})+3))^2 - 4}}{2}.$$

- Both $y_\pm$ have sometimes absolute value > 1
- $\mathrm{m}(P_{\sqrt{7},3}) = \frac{\log 7}{2}$.

Solution: Instead of

$$\mathrm{m}(P_{\sqrt{7},3}) = \mathrm{m}(y - y_+(x)) + \mathrm{m}(y - y_-(x))$$

look at

$$\mathrm{m}(y - y_+(x)) \stackrel{?}{=} -\frac{1}{2}L'(E_{21},0) + \frac{1}{8}\log 7$$

$$\mathrm{m}(y - y_-(x)) \stackrel{?}{=} \frac{1}{2}L'(E_{21},0) + \frac{3}{8}\log 7$$

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Strategies for the proof

Fix

$$c = \frac{\sqrt{2}(a^2-1)}{\sqrt{a^2+1}}$$

1. Prove

$$\begin{aligned}\mathrm{m}(P_{1,4(a^2-1)/(a^2+1)}) &= 4\mathrm{m}(y - y_-(x)) - 3\log a\\ &= -4\mathrm{m}(y-y_+(x)) + \log a.\end{aligned}$$

   By differentiating respect to a. Then set $a = \sqrt{7}$.

2. Prove

$$4\mathrm{m}(y-y_-(x)) = -4\mathrm{m}(y-y_+(x)) + p\log 7, \qquad p \in \mathbb{Z}$$

   by working with the regulator.

Isogeny and regulator

$\exists q \in (1/8)\mathbb{Z}$,

$$\mathrm{m}(y - y_+(x)) = \frac{1}{2\pi}\int_{\gamma_+} \eta(x,y) = \frac{1}{8\pi}\int_{\gamma'} \eta(x',y') + q \log a,$$
$$-\mathrm{m}(y - y_-(x)) = \frac{1}{2\pi}\int_{\gamma_-} \eta(x,y) = \frac{1}{8\pi}\int_{\gamma'} \eta(x',y') + (q-1) \log a.$$

We work with the isogeny and the elliptic dilogarithm D^E.

- The divisors of x, y, x', y' are supported in torsion points of order at most 4.
- The boundaries of $\gamma_\pm, \gamma'$ are supported in torsion points of order at most 8.

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The final step

Prove

$$\mathrm{m}(y - y_-(x)) = \frac{1}{2}L'(E_{21}, 0) + \frac{3}{8}\log 7.$$

The tame symbols are not trivial. $\int_\gamma \eta(x, y)$ is not a function on $K_2(E) \times H_1(E, \mathbb{Z}))$.

Solution: avoid integrating around the problematic points of $X_0(21)$.

$$\mathrm{m}(y - y_-(x)) = \frac{1}{2\pi}\int_{2/7}^{i\infty} \eta(\sqrt{7}\widetilde{x}(\tau), \widetilde{y}(\tau))$$

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Other directions

- Other families (results by Mellit, Brunault, Rogers & Zudilin, Bertin)
- Higher genus (results by Brunault, Bertin & Zudilin)
- Three variables and $L'(E,-1)$ (relations between formulas, Boyd, L.)
- Predict s_k.

Thank you!