Salem numbers of trace -2 and a conjecture of Estes and Guralnick

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- m_A is totally real
- m_A is separable

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BUT. . .

Counterexamples to EG

- Small discriminant. (Dobrowolski, 2008)
  - If irreducible $f = m_A$ for some A, and f has degree n, then there exists a totally real, degree $n-1$, monic $g(x) \in \mathbb{Z}[x]$ whose roots interlace with those of f.

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    CONVENTION: all polynomials are in $\mathbb{Z}[x]$ and monic

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  - The existence of such a g implies that the discriminant of f is at least n^n.
  - Consider the minimal polynomial of $e^{2\pi i/n} + e^{-2\pi i/n}$ for highly composite n.

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  - Degrees 6, 7, . . . , 15.

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  – Any irreducible totally positive f of degree n and trace $< 2n - 1$.

Counterexamples to EG

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- Small numbers of interlacing polynomials. (Y)
- Small span. (M, 2010)
- Small trace. (M,Y, 2014)
  - e.g., (M., Smyth, 2004) $x^{10} - 18x^9 + 134x^8 - 537x^7 + 1265x^6 - 1798x^5 + 1526x^4 - 743x^3 + 194x^2 - 24x + 1$

Counterexamples to EG

- Small discriminant. (Dobrowolski, 2008)
- Small numbers of interlacing polynomials. (Y)
- Small span. (M, 2010)
- Small trace. (M,Y, 2014)
  - (M,Y, 2015) All degrees ≥ 12.

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Proof

Step 1: degree ≥ 42.

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Proof

Step 1: degree ≥ 42.

Step 2: degrees 24, 26, . . . , 40.

An infinite family of Salem numbers of trace -2 (M,Smyth, 2004)

Put

$$\frac{q}{p} = \frac{z^5-1}{(z^2-1)(z^3-1)} + \frac{z^{12}-1}{(z^5-1)(z^7-1)} + \frac{z^{11+n}-1}{(z^{11}-1)(z^n-1)},$$

where $\gcd(n, 2\times 3\times 5\times 7\times 11) = 1$, $n > 1$.

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Then $(z^2 - 1)p - zq$ is the minimal polynomial of a Salem number, degree $n + 25$, trace -2.

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Then $(z^2 - 1)p - zq$ is the minimal polynomial of a Salem number, degree $n + 25$, trace -2.

This gives examples of Salem numbers of trace -2 of every sufficiently large degree $n + 25$, subject to $\gcd(n, 2 \times 3 \times 5 \times 7 \times 11) = 1$.

Salem numbers of trace -2 for all degrees ≥ 42

Replace $(2, 3, 5, 7, 11)$ in the above by some other 5-tuples of primes (not every 5-tuple works).

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Replace $(2, 3, 5, 7, 11)$ in the above by some other 5-tuples of primes (not every 5-tuple works).

Find fifteen 5-tuples, using primes up to 19, that cover all even degrees ≥ 42.

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- Thank you.