University of Basel

On Narkiewicz's Property (P)

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The Geometry, Algebra and Analysis of Algebraic Numbers at BIRS

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Notations

- $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$. All algebraic extensions of $\mathbb{Q}$ are contained in $\overline{\mathbb{Q}}$.
- K will always denote a number field.
- h is the absolute logarithmic Weil-height on $\mathbb{P}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \infty$.
- ζ_n is a prinitive n-th root of unity.
- $\mathbb{Q}^{(d)}$ is the compositum of all number fields of degree $\leq d$.

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Property (P)

Definition

A field F has property (P) if and only if there is no infinite subset $X \subseteq F$, such that $f(X) = X$ for any polynomial $f \in F[x]$ with $\deg(f) \geq 2$.

Examples

- Every finite field has property (P). (trivial)
- No algebraically closed field has property (P).
- Every finitely generated field has property (P). (Narkiewicz 1962)

From now on we will focus on subfields of $\overline{\mathbb{Q}}$.

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An example of Dvornicich and Zannier

Set $F' = \mathbb{Q}(\{\zeta_{7^n} | n \in \mathbb{N}\})$, and let F be the unique subfield of F' such that $[F' : F] = 3$.

Then F has property (P).

Note that F' does not have property (P), as we can set

- $X = \{\zeta_{7^n}^m | n, m \in \mathbb{N}\} \subseteq F'$ $|X| = \infty$
- $f(x) = x^7$ $\deg(f) \geq 2$

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The conjecture

Conjecture (Narkiewicz 1963)

Let d be a positive integer, then $\mathbb{Q}^{(d)}$ has property (P).

First partial results to this conjecture:

- $d = 1$: Narkiewicz 1962
- $d = 2$: follows from work of Bombieri and Zannier 2001

In the following we will prove this conjecture in general.

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In the following we will prove this conjecture in general.

Canonical heights

Theorem (Call, Silverman 1993)

Let $f \in K(x)$ be a rational map with $\deg(f) \geq 2$. Then there is exactly one function $\widehat{h}_f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$, such that

$$\widehat{h}_f \circ f = \deg(f)\widehat{h}_f \quad \text{and} \quad |\widehat{h}_f - h| = O_f(1)$$

Facts

- $\widehat{h}_f(\alpha) = 0 \Leftrightarrow \alpha \in \mathrm{PrePer}(f) \Leftrightarrow \{\alpha, f(\alpha), f(f(\alpha)), \dots\}$ is finite.
- $h = \widehat{h}_{x^2}$

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Julia sets

Let v be a place of K, and as usual let $\mathbb{C}_v$ be the completion of the algebraic closure of K_v. We use the topology induced by v on $\mathbb{P}^1(\mathbb{C}_v)$.

Then the v-adic Fatou set of a rational map $f \in K(x)$ is the maximal open subset $\mathcal{F}_v(f)$ of $\mathbb{P}^1(\mathbb{C}_v)$ where the family $\{f^n | n \in \mathbb{N}\}$ is equicontinuous at every point in $\mathcal{F}_v(f)$.

The v-adic Julia set of f is given by $\mathcal{J}_v(f) = \mathbb{P}^1(\mathbb{C}_v) \setminus \mathcal{F}_v(f)$.

Julia sets

Slogan:

"$\mathcal{J}_v(f) \subseteq \mathbb{P}^1(\mathbb{C}_v)$ is the subset where iteration of f acts chaotically."

Facts (For $f \in K(x)$ arbitrary we have:)

- $f(\mathcal{J}_v(f)) = f^{-1}(\mathcal{J}_v(f)) = \mathcal{J}_v(f)$
- $\mathcal{J}_v(f) \neq \emptyset$ for all places $v \mid \infty$ on K
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Main Theorem

Let K^{tv} be the maximal totally v-adic field extension of K. That is, the maximal Galois extension of K which can be embedded in K_v.

Theorem (Fili / Miner (slightly weaker version), P.)

Let $f \in K(x)$ be a rational map with $\deg(f) \geq 2$, and let v be a place on the number field K. Then the following statements are equivalent:

1. $J_v(f) \not\subset \mathbb{P}^1(K_v)$ or $J_v(f) = \emptyset$.
2. $|\text{PrePer}(f) \cap \mathbb{P}^1(K^{tv})| < \infty$.
3. $\liminf_{\alpha \in K^{tv}} \widehat{h}_f(\alpha) > 0$.

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The equidistribution theorem

The main ingredient for the proof is an equidistribution theorem for points of small height on the v-adic Berkovich line $\mathbb{P}_v^{\mathsf{Berk}}$. This is due to Baker / Rumely, Favre / Rivera-Letelier, Chambert-Loir ($\sim$ 2006), Yuan (2008), ...

The equidistribution theorem

Let $f \in K(x)$ and v be as in the theorem. There is a unique Borel probability measure $\mu_{v,f}$ on $\mathbb{P}_v^{\text{Berk}}$ such that

$$f_* \mu_{v,f} = \mu_{v,f} \quad \text{and} \quad f^* \mu_{v,f} = \deg(f) \mu_{v,f}.$$

For an $\alpha \in \overline{\mathbb{Q}}$ we define the probability measure

$$\overline{\delta_\alpha} = \frac{1}{[K(\alpha) : K]} \sum_{z \text{ is } K\text{-conj. to } \alpha} \delta_z$$

on $\mathbb{P}(\mathbb{C}_v) \subset \mathbb{P}_v^{\text{Berk}}$, where δ_z is the Dirac measure at the point z. Let $\alpha_1, \alpha_2, \dots \in \overline{\mathbb{Q}}$ be pairwise distinct with $\widehat{h}_f(\alpha_i) \to 0$, as $i \to \infty$. Then we have

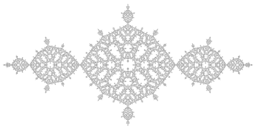
$$\overline{\delta_{\alpha_i}} \xrightarrow{\text{weakly}} \mu_{v,f}.$$

Descriptive version of equidistribution

Let $\alpha_1, \alpha_2, \dots \in \overline{\mathbb{Q}}$ be pairwise distinct with $\widehat{h}_f(\alpha_i) \to 0$, as $i \to \infty$. Then $\{\sigma(\alpha_i) | \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)\} \longrightarrow \mathcal{J}_v(f)$.

Example ($K = \mathbb{Q}$, $v = \infty$, $f = x^2 + \frac{1}{500x^2} - 1$)

Then $K^{tv} = \mathbb{Q}^{tr}$ is the maximal totally real field extension of $\mathbb{Q}$. Assume there is a sequence $\alpha_1, \alpha_2, \dots \in \mathbb{Q}^{tr}$, with $\widehat{h}_f(\alpha_i) \to 0$. Then $\{\sigma(\alpha_i) | \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\} \subseteq \mathbb{R}$, and hence cannot accumulate around the beautiful Julia set of f which looks like



This is a contradiction to the equidistribution theorem, and hence we have $\liminf_{\alpha \in \mathbb{Q}^{tr}} \widehat{h}_f(\alpha) > 0$.

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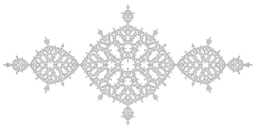

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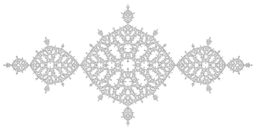

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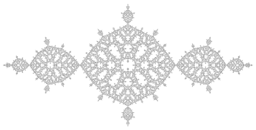

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Corollary

Let F/K be a Galois extension with $[F_w : K_v] < \infty$ for infinitely many places v on K. Then we have

$$\liminf_{\alpha \in F} \widehat{h}_f(\alpha) > 0 \text{ for all } f \in F(x), \text{ with } \deg(f) \geq 2. \qquad (\star)$$

Proof: Let $f \in F(x)$, with $\deg(f) \geq 2$, be arbitrary. There must be a (necessarily finite) place v on K with $[F_w : K_v] < \infty$ and $J_v(f) = \emptyset$. By enlarging K we may assume $[F_w : K_v] = 1$; i.e. $F \subseteq K^{tv}$. The main theorem now states

$$\liminf_{\alpha \in F} \widehat{h}_f(\alpha) \geq \liminf_{\alpha \in K^{tv}} \widehat{h}_f(\alpha) > 0.$$

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Definition

A field F which satisfies statement $(\star)$ is called to have the **universal strong Bogomolov property**, or property (USB).

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Connection to property (P)

Let $f(x) \in K[x]$ be a polynomial and $X \subseteq \overline{\mathbb{Q}}$ a set with $f(X) = X$.
Then X consists of a union of

closed cycles:

infinite orbits: $\dots \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \dots$

Both yields points of small $\widehat{h}_f$-height!

This immediately gives us $(USB) \Rightarrow (P)$.

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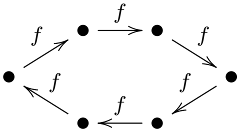

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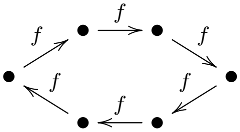

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Proof of Narkiewicz's conjecture

Theorem

The field $\mathbb{Q}^{(d)}$ has property (USB), and hence property (P), for any positive integer d.

Proof: By the corollary, it suffices to notice that for infinitely many primes p we have $[(\mathbb{Q}^{(d)})_w : \mathbb{Q}_p] < \infty$. But this is true for all primes p, since there are only finitely many field extensions of $\mathbb{Q}_p$ of fixed degree. Hence $\mathbb{Q}^{(d)}$ has the properties (USB) and (P). $\square$

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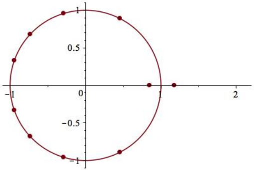
Thanks for your attention!

Salem numbers

A Salem number α is a real positive algebraic integer of degree at least 4 with all but one conjugate lying on the unit circle. The map $z \mapsto z + z^{-1}$ transforms a Salem number to a totally real number.



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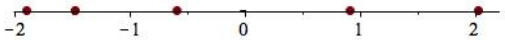

The height of a Salem number is $h(\alpha) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \log(\alpha)$.

Chebyshev polynomials

The polynomial $f_{-2}(x) = x^2 - 2$ is the second Chebyshev polynomial and satisfies

$$f_{-2}(z + z^{-1}) = z^2 + z^{-2} \quad \text{for all } z \in \mathbb{C}^*.$$

This implies

$$h(\alpha) = \frac{1}{2}\widehat{h}_{f_{-2}}(\alpha + \alpha^{-1}) \quad \text{for all } \alpha \in \overline{\mathbb{Q}}.$$

Lemma

If there is a constant $c > 0$ such that $\widehat{h}_{f_{-2}}(\alpha) \geq \frac{c}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$ for all $\alpha \in \mathbb{Q}^{tr} \setminus \mathrm{PrePer}(f_{-2})$, then every Salem number is $\geq e^{c/2} > 0$.

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Lemma

If there is a constant $c > 0$ such that $\widehat{h}_{f_{-2}}(\alpha) \geq \frac{c}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$ for all $\alpha \in \mathbb{Q}^{tr} \setminus \mathrm{PrePer}(f_{-2})$, then every Salem number is $\geq e^{c/2} > 0$.

weak support

An easy application of the main theorem is the following.

Lemma

Let $f_c(x) = x^2 + c \in \mathbb{Q}[x]$. Then we have

$$\liminf_{\alpha \in \mathbb{Q}^{tr}} \widehat{h}_{f_c}(\alpha) > 0 \quad \Longleftrightarrow \quad c > -2$$

What happens at $c = -2$ (in the moduli space of quadratic polynomials)? Can we use (effective) potential theory on the Berkovich line as in Baker / Rumely (2006), Favre / Rivera-Latelier (2007), Fili / P. (2015) to achieve this Lehmer type bound? ...

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