A Survey of Arithmetic Applications of Capacity Theory

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The Geometry, Algebra and Analysis of Algebraic Numbers
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The logarithmic capacity $\gamma(E)$

For a compact set $E \subset \mathbb{C}$, the logarithmic capacity $\gamma_\infty(E)$ is a measure of size arising in potential theory, with applications in analysis, probability, approximation theory, and arithmetic.

In analysis, the main distinction is between sets of capacity 0 and sets of positive capacity. Sets of capacity 0 are extremely small. Finite sets and compact countable sets have capacity 0. Sets of capacity 0 have Lebesgue measure 0, but not all compact sets of Lebesgue measure 0 have capacity 0; the middle thirds Cantor set has positive capacity.

Sets of capacity 0 tend to be "invisible" to holomorphic and harmonic functions. Many results which hold for isolated points, like the Riemann Extension Theorem, remain valid for sets of capacity 0.

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Examples of capacities

In arithmetic, the main distinctions are between sets of capacity < 1, $= 1$, and > 1. There is often a sharp break in arithmetic phenomena at capacity 1.

Here are some examples of capacities:

When $E = D(a, r)$ is a disc, $\gamma_\infty(E) = r$.

When $E = [a, b] \subset \mathbb{R}$ is a segment, $\gamma_\infty(E) = (b - a)/4$.

When $E = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$ is an ellipse, $\gamma_\infty(E) = (a + b)/2$.

For Mandelbrot set, $\gamma_\infty(\mathcal{M}) = 1$.

For the Julia set of a polynomial $a_0 + \cdots + a_d z^d$ with $d \geq 2$,

$$\gamma_\infty(\mathcal{J}) = |a_d|^{-1/(d-1)}$$

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Green's functions

The best way to determine $\gamma_\infty(E)$ is to guess its Green's function $G(z, \infty; E)$:

When $\gamma_\infty(E) > 0$, the Green's function is the unique function which satisfies

- $G(z, \infty; E) = 0$ for $z \in E$;
- $G(z, \infty; E)$ is continuous on $\mathbb{C}$, except possibly on a set of capacity 0 contained in ∂E;
- $G(z, \infty; E)$ is harmonic in $\mathbb{C} \backslash E$;
- There is a constant $V_\infty(E)$ such that $G(z, \infty; E) = \log(|z|) + V_\infty(E) + o(1)$ when $z \to \infty$.

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$$\gamma_\infty(E) = e^{-V_\infty(E)} .$$

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$$G(z, \infty; E) = \log^+\left(\left|\frac{\sqrt{(z-a)/z-b)}+1}{\sqrt{(z-a)/z-b)}-1}\right|\right).$$

This is gotten by conformally mapping $\mathbb{C} \backslash [a, b]$ to $\mathbb{C} \backslash D(0, 1)$.

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The Polýa-Carlson Theorem

The earliest arithmetic application of capacity theory was given by F. Carlson and G. Pol'ya:

Theorem (Carlson 1921, extended by Pol'ya in 1922 and 1928)

Let $f(z) = \sum_{n=k}^{\infty} a_n z^{-n} \in \mathbb{Z}[z]$ *be a Laurent series with integer coefficients, which converges in a neighborhood of* ∞*, and which has an analytic continuation to the complement of a set* $E \subset \mathbb{C}$ *with* $\gamma(E) < 1$*. Then* $f(z)$ *is the expansion of a rational function* $p(z)/q(z) \in \mathbb{Q}(z)$*.*

For example, the series $f(z) = 2z^{-3} + 4z^{-5} + 6z^{-7} + \cdots$ converges near ∞ and has an analytic continuation to the complement of the segment $[-1, 1]$. It is the Laurent expansion of $F(z) = 2z/(1 - z^2)^2$.

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In 1968 Raphael Robinson gave a partial converse to the Pol'ya-Carlson theorem:

Theorem (Robinson, 1968)

Let $E \subset \mathbb{C}$ be a compact set, stable under complex conjugation, with $\gamma(E) > 1$. Then there are infinitely many Laurent series $f(z) = \sum_{n=k}^{\infty} a_n z^{-n} \in \mathbb{Z}[z]$ which converge near ∞ and have an analytic continuation to the complement of E, which are not rational.

There is also a theorem of Pommerenke, which I could not locate while preparing these slides, which says that the Pol'ya-Carlson theorem does hold when $\gamma(E) = 1$, under suitable conditions on the Laurent series and the boundary ∂E. (?? continuous extension to ∂E, ∂E in $\mathrm{Lip}(1/2)$, E connected)

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Fekete's Theorem

Another early arithmetic application of capacity was given by M. Fekete:

Theorem (Fekete, 1923)

Let $E \subset \mathbb{C}$ be a compact set which is stable under complex conjugation. If $\gamma_\infty(E) < 1$, there are only finitely many algebraic integers whose conjugates all belong to E.

For example, if $E = D(0, r)$ with $0 < r < 1$, so $\gamma_\infty(E) = r$, then 0 is the only algebraic integer in E.

If $E = [-1/2, 1/2]$, so $\gamma_\infty(E) = 1/4$, again 0 is the only algebraic integer with all its conjugates in E.
If $E = [-1, 1]$, so $\gamma_\infty(E) = 1/2$, then $-1, 0, 1$ are the only algebraic integers with all their conjugates in E.

If $E = [-2r, 2r]$ where $0 < r < 1$, so $\gamma_\infty(E) = r$, the only algebraic integers with all conjugates in E are the finitely many roots of Chebyshev polynomials with $|2\cos(2\pi k/n)| \le 2r$.

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The Fekete-Szegö Theorem

The converse to Fekete's theorem was found by Fekete and G. Szegö more than 30 years later.

Theorem (Fekete, 1923; Fekete-Szegö, 1955)

Let $E \subset \mathbb{C}$ be a compact set which is stable under complex conjugation. If $\gamma(E) < 1$, there is a neighborhood $U \supset E$ which contains only finitely many complete conjugate sets of algebraic integers. If $\gamma(E) \geq 1$, then every neighborhood $U \supset E$ contains infinitely many conjugate sets of algebraic integers.

Note that this combines a finiteness theorem and an existence theorem.

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Strengthening the hypotheses on E, one can require that the conjugate sets of algebraic integers belong to E itself:

Theorem (Fekete-Szegö, 1955)

Let $E \subset \mathbb{C}$ be a compact set which is stable under complex conjugation, has a piecewise smooth boundary, and is the closure of its interior. If $\gamma(E) > 1$, there are infinitely many conjugate sets of algebraic integers in E.

For example, for the disc $D(0, r)$, when $r \geq 1$ then all the roots of unity belong to $D(0, 1) \subset D(0, r)$.

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What about sets $E \subset \mathbb{R}$?
In a significant advance, Raphael Raphael proved

Theorem (Robinson, 1964)

Let $E \subset \mathbb{R}$ be a finite union of closed intervals. If $\gamma(E) > 1$, E contains infinitely many conjugate sets of algebraic integers.

For example, when $E = [-2r, 2r]$ with $r \geq 1$, the roots of all the Chebyshev polynomials belong to $[-2, 2] \subset E$.

However, when $\gamma(E) = 1$ it is rare (even for intervals $[a-2, a+2]$ with arbitrary $a \in \mathbb{R}$) and in general it is a hard open problem to determine when E contains infinitely many conjugate sets of algebraic integers.

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Uniform Approximation by Polynomials with Integer Coefficients

A third application of capacities involves approximation of continuous functions on an interval by polynomials with *integer coefficients*.

Before stating the theorem, let us consider some examples. Let $E = [-1/2, 1/2]$ and let $f : E \to \mathbb{R}$ be continuous. Weierstrass's theorem says f can be uniformly approximated by polynomials in $\mathbb{R}[x]$. Suppose however, that we had uniform approximation by polynomials $P_1(z), P_2(x), \ldots \in \mathbb{Z}[x]$. Since each polynomial takes an integer value at $x = 0$, and since the values $P_1(0), P_2(0), \cdots$ converge uniformly to $f(0)$, it must be that $f(0) \in \mathbb{Z}$ and that $P_n(0) = f(0)$ for all large n.

Theorem (Integer Polynomial Approximation on [-1/2,1/2)

.] If $f : [-1/2, 1/2] \to \mathbb{R}$ is continuous, then f can be uniformly approximated by polynomials in $\mathbb{Z}[x]$ if and only if $f(0) \in \mathbb{Z}$.

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Uniform Approximation by Polynomials with Integer Coefficients

Next suppose $E = [-1, 1]$. By similar considerations, one sees that necessarily $f(-1), f(0), f(1) \in \mathbb{Z}$. Furthermore, for any $P(z) \in \mathbb{Z}[x]$, one has $P(-1) \equiv P(1) \pmod 2$. Hence it must be that $f(-1) \equiv f(1) \pmod 2$ as well.

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Uniform Approximation by Polynomials with Integer Coefficients

The general theorem is as follows:

Theorem (?L. Ferguson)

Let $E = [a, b] \subset \mathbb{R}$, and let $\mathcal{C}([a, b])$ be the space of continuous, real-valued function on E. Then:

(A) If $\gamma(E) < 1$, let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset E$ be the finite set of algebraic integers whose conjugates all belong to E, given by Fekete's theorem. Let $f \in \mathcal{C}([a, b])$. Then f can be uniformly approximated by polynomials in $\mathbb{Z}[x]$ if and only if its values at the points in A are matchable by those of a polynomial in $\mathbb{Z}[x]$: there is a $P(x) \in \mathbb{Z}[x]$ such that $f(\alpha) = P(\alpha)$ for each $\alpha \in A$.

(B) If $\gamma(E) > 1$, then $\mathbb{Z}[x]$ is discrete in $\mathcal{C}([a, b])$ for the L^∞ norm. Hence there are functions $f \in \mathcal{C}([a, b])$ which cannot be uniformly approximated by polynomials in $\mathbb{Z}[x]$.

Bilu's Equidistribution Theorem

A more recent application of capacity theory is Bilu's theorem.

Let $h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ be the absolute Weil height, which has the local decomposition

$$h(\alpha) = \sum_{\text{all places } p \text{ of } \mathbb{Q}} \frac{1}{[\mathbb{Q}(\alpha) : \mathbb{Q}]} \sum_{\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}_p} \log^+(|\sigma(\alpha)|)_p) .$$

Let $\mu = d\theta/(2\pi)$ be the uniform measure of mass 1 on the unit circle $C(0,1)$, viewed as a singular measure on $\mathbb{C}$.

Theorem (Bilu, 1997)

Let $\alpha_1, \alpha_2, \alpha_3, \cdots \in \overline{\mathbb{Q}}$ be a sequence of numbers for which $[\mathbb{Q}(\alpha_n) : \mathbb{Q}] \to \infty$ and $h(\alpha_n) \to 0$. For each n, let δ_n be the discrete probability measure supported equally on the conjugates of α_n. Then the measures δ_n converge weakly to μ as $n \to \infty$.

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Bilu's Equidistribution Theorem

Although Bilu's original proof used Fourier analysis, a small observation suggests that the theorem has a capacity-theoretic nature: $\log^+(|z|)$ is the Green's function of the unit disc $D(0,1) \subset \mathbb{C}$.

Let $E \subset \mathbb{C}$ be an arbitrary compact set, stable under complex conjugation, with capacity 1. Let $h_E : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ be the "height attuned to E" gotten by replacing each archimedean term $\log^+(|\sigma(\alpha)|)$ in the decomposition of $h(\alpha)$ with $G(\sigma(\alpha), \infty; E)$:

$$h_E(\alpha) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{\sigma:\mathbb{Q}(\alpha)\hookrightarrow\mathbb{C}} G(\sigma(\alpha), \infty; E)$$

$$+ \sum_{\text{finite places } p \text{ of } \mathbb{Q}} \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{\sigma:\mathbb{Q}(\alpha)\hookrightarrow\mathbb{C}_p} \log^+(|\sigma(\alpha)|)_p \,.$$

The distributional Laplacian of $G(z, \infty; E)$ is a probability measure supported on $\partial E \subset \mathbb{C}$.

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Theorem (R., 1999)

Let $E \subset \mathbb{C}$ *be a compact set of capacity* 1*, stable under complex conjugation. Let* $\alpha_1, \alpha_2, \alpha_3, \cdots \in \overline{\mathbb{Q}}$ *be a sequence of numbers for which* $[\mathbb{Q}(\alpha_n) : \mathbb{Q}] \to \infty$ *and* $h_E(\alpha_n) \to 0$*. For each* n*, let* δ_n *be the discrete probability measure supported equally on the conjugates of* α_n*. Then the measures* δ_n *converge weakly to* μ *as* $n \to \infty$*.*

Towards a Modern Theory of Capacities

We will now skip over a considerable amount of history, and present the modern, multi-center adelic theory of capacities. For each of the theorems above, there is an adelic version which holds on any smooth projective curve over a global field.

It is good to regard a set $E \subset \mathbb{C}$ as belonging to $\mathbb{P}^1(\mathbb{C})$. For an arbitrary $\zeta \in \mathbb{P}^1(\mathbb{C}) \backslash E$, the Green's function $G(z, \zeta; E)$ can be defined, with properties as before but relative to ζ.

Fixing a uniformizer $z - \zeta$, one defines the Robin constant relative to ζ by

$$V_\zeta(E) = \lim_{z \to \zeta}(G(z, \zeta; E) + \log(|z - \zeta|)),$$

and the capacity relative to ζ by

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One also has capacities and Green's functions for sets in $\mathbb{P}^1(\mathbb{C}_p)$,

For a disc $D_p(a,r) = \{z \in \mathbb{C}_p : |z-a|_b \le r\} \subset \mathbb{C}_p$,

$$G(z,\infty;E) = \log^+(|z-a|_p/r),$$
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For the set of p-adic integers $\mathbb{Z}_p$, if μ_p is Haar measure on $\mathbb{Z}_p$,

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With some work, for any smooth complete curve $\mathcal{C}/\mathbb{C}_p$, one can define Green's functions and capacities on $\mathcal{C}(\mathbb{C}_p)$.

The Single-center Adelic Capacity

For each place p of $\mathbb{Q}$, let $E_p \subset \mathbb{C}_p$ be a bounded closed set, stable under the group of continuous automorphisms $\mathrm{Aut}^c(\mathbb{C}_p/\mathbb{Q}_p) \cong \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. For all but finitely many p, assume that $E_p = D_p(0,1) = \{z \in \mathbb{C}_p : |z|_p \leq 1\}$. Put

$$\mathbb{E} = \prod_{\text{all places } p \text{ of } \mathbb{Q}} \gamma_{p,\infty}(E_p)$$

Define the global capacity $\gamma(\mathbb{E},\infty) = \prod_p \gamma_{p,\infty}(E_p)$.

Example

For example, fix a finite set of primes S, let $E_\infty = [a, b]$ at the archimedean place, and let $E_p = \mathbb{Z}_p$ for each p. Then

$$\gamma(\mathbb{E}, \infty) = \frac{b-a}{4} \cdot \prod_{p \in S} p^{-1/(p-1)} .$$

Theorem (An Adelic Fekete/Fekete-Szegö Theorem)

With $\mathbb{E}$ as above,
(A) If $\gamma(\mathbb{E}, \infty) < 1$ there are only finitely many points of $\overline{\mathbb{Q}}$ (necessarily algebraic integers) whose archimedean conjugates belong to $[a, b]$, whose p-adic conjugates belong to $\mathbb{Z}_p$ for each $p \in S$, and whose p-adic conjugates for $p \notin S$ belong to $D_p(0, 1)$.
(B) If $\gamma(\mathbb{E}, \infty) > 1$ there are infinitely many.

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Capacities relative to multiple centers

What about the conjugates in $\mathbb{C}_p$ for $p \notin S$?

They all belong to $D_p(0,1) = \{z \in \mathbb{C}_p : |z|_p \le 1\}$.

An algebraic number is an algebraic integer if and only if its p-adic conjugates belong to $D_p(0,1)$ for all p.

Another way of viewing the integrality condition is to say that the conjugates *avoid* ∞ in $\mathbb{P}^1(\mathbb{C}_p)$ for all finite primes. Note that $D_p(0,1) = \mathbb{P}^1(\mathbb{C}_p) \backslash B(\infty, 1)^-$.

By allowing more general sets at nonarchimedean places, one can construct algebraic numbers which satisfy prescribed conditions at finitely many places, and are integral at the remaining places.

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Example

Up to now we have discussed numbers whose conjugates "avoid ∞"; numbers which "avoid" other points are also interesting.

An algebraic number is a unit if it belongs to $U_p = \{z \in \mathbb{C}_p : |z|_p = 1\}$, for each finite place p. Alternately, it "avoids" both 0 and ∞ at all finite places.

Theorem (Robinson, 1968)

Let $0 < a < b \in \mathbb{R}$. Then the interval $[a, b]$ contains infinitely many totally real algebraic units if and only if

1. $\log(\frac{b-a}{4}) > 0$ *and*
2. $\log(\frac{b-a}{4}) \cdot \log(\frac{b-a}{4ab}) - \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}})^2 > 0$

If either condition fails, there are only finitely many.

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Discussion

An algebraic number is a unit if and only if its conjugates belong to

$$D_p(0,1)\backslash D_p(0,1)^- \;=\; \mathbb{P}^1(\mathbb{C}_p)\backslash(B(\infty,1)^- \cup B(0,1)^-)$$

for each p, that is, if it avoids ∞ and 0 at each finite place.

The conditions in the Theorem are equivalent to the negative definiteness of

$$\Gamma \;=\; \begin{pmatrix} -\log(\frac{b-a}{4}) & \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) \\ \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) & -\log(\frac{b-a}{4ab}) \end{pmatrix} .$$

There are Green's functions and Robin constants with respect any point not in E. Here

$$\Gamma \;=\; \Gamma(E,\{\infty,0\}) \;=\; \begin{pmatrix} V_\infty(E) & G(0,\infty;E) \\ G(\infty,0;E) & V_0(E) \end{pmatrix}$$

is the 'Green's matrix of E' with respect to ∞ and 0.

Discussion

An algebraic number is a unit if and only if its conjugates belong to

$$D_p(0,1)\backslash D_p(0,1)^- \;=\; \mathbb{P}^1(\mathbb{C}_p)\backslash(B(\infty,1)^- \cup B(0,1)^-)$$

for each p, that is, if it avoids ∞ and 0 at each finite place.

The conditions in the Theorem are equivalent to the negative definiteness of

$$\Gamma \;=\; \begin{pmatrix} -\log(\frac{b-a}{4}) & \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) \\ \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) & -\log(\frac{b-a}{4ab}) \;. \end{pmatrix}$$

There are Green's functions and Robin constants with respect any point not in E. Here

$$\Gamma \;=\; \Gamma(E,\{\infty,0\}) \;=\; \begin{pmatrix} V_\infty(E) & G(0,\infty;E) \\ G(\infty,0;E) & V_0(E) \end{pmatrix}$$

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The General Framework

Let K be a global field, a number field or a finite extension of $\mathbb{F}_p(t)$ for some p. Fix an algebraic closure $\widetilde{K}$ of K.

Let $\mathcal{C}/K$ be a smooth, projective, geometrically integral curve.

Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite, galois-stable set of points: the points to *avoid*.

For each place v of K, let $E_v \subset \mathcal{C}(\mathbb{C}_v)$ be a nonempty set disjoint from $\mathfrak{X}$. We will require that E_v be galois-stable, and that it be a finite union of 'v-basic sets' as defined below.

For all but finitely many places, we require that $E_v = \mathcal{C}(\mathbb{C}_v) \backslash (\bigcup_{i=1}^m B(x_i, 1)^-)$ be '$\mathfrak{X}$-trivial'.

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The Cantor Capacity

For each place v, define the local Green's matrix to be the $m \times m$ symmetric matrix

$$\Gamma(E_v, \mathfrak{X}) = \begin{pmatrix} V_{x_1}(E_v) & G(x_2, x_1; E_v) & \cdots & G(x_m, x_1; E_v) \\ G(x_1, x_2; E_v) & V_{x_2}(E_v) & \cdots & G(x_m, x_2; E_v) \\ \vdots & \vdots & \ddots & \vdots \\ G(x_1, x_m; E_v) & G(x_2, x_m; E_v) & \cdots & V_{x_m}(E_v) \end{pmatrix}$$

If $\mathfrak{X} \subset \mathcal{C}(K)$, put $\mathbb{E} = \prod_v E_v$. Define the global Green's matrix

$$\Gamma(\mathbb{E}, \mathfrak{X}) = \sum_v \Gamma(E_v, \mathfrak{X}) \log(Nv) ,$$

where Nv is the order of the residue field at v, and $\log(Nv) = 1$ if $K_v \cong \mathbb{R}$ and $\log(Nv) = 2$ if $K_v \cong \mathbb{C}$.

If $\mathfrak{X} \not\subset \mathcal{C}(K)$, put $L = K(\mathfrak{X})$ and let $\Gamma(\mathbb{E}, \mathfrak{X}) = \frac{1}{[L:K]}\Gamma(\mathbb{E}_L, \mathfrak{X})$.

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$\mathcal{P}_m = \{(s_1, \ldots, s_m) \in \mathbb{R}^m : s_1, \ldots, s_m \geq 0, s_1 + \cdots + s_m = 1\}$
denote the set of m-element *probability vectors*.

There is a simple criterion for a symmetric $m \times m$ matrix to be negative definite: The *value of* Γ *as a matrix game* is

$$\mathrm{val}(\Gamma) \ = \ \max_{\vec{s} \in \mathcal{P}_m} \min_{\vec{r} \in \mathcal{P}_m} {}^t\vec{s}\,\Gamma\,\vec{r} \ ,$$

and Γ is negative definite if and only if $\mathrm{val}(\Gamma) < 0$.

In general, for $\mathbb{E} = \prod_v E_v$ and $\mathfrak{X} = \{x_1, \ldots, x_m\}$, the Cantor capacity of $\mathbb{E}$ with respect to $\mathfrak{X}$ is defined to be

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The Fekete-Szegö Theorem with Local Rationality Conditions

Theorem (R, 2012)

Let K be a global field. Let $\mathcal{C}/K$ be a smooth, projective, geometrically integral curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite set of points stable under $\mathrm{Aut}(\widetilde{L}/K)$. For each place v of K, let $E_v \subset \mathcal{C}(\mathbb{C}_v) \backslash \mathfrak{X}$ be a nonempty set which is a finite union of v-basic sets and is stable under the group of continuous automorphisms $\mathrm{Aut}^c(\mathbb{C}_v/K_v) \cong \mathrm{Aut}(\widetilde{K}^{\mathrm{sep}}/K_v)$. Assume that E_v is $\mathfrak{X}$-trivial for all but finitely many v.

Put $\mathbb{E} = \prod_v E_v$. If $\gamma(\mathbb{E}, \mathfrak{X}) > 1$, there are infinitely many points of $\mathcal{C}(\widetilde{K})$ whose conjugates in $\mathcal{C}(\mathbb{C}_v)$ all belong to E_v, for each place v of K. If $\gamma(\mathbb{E}, \mathfrak{X}) < 1$, there are only finitely many such points.

Multi-center Adelic Versions of other Classical Theorems

There are multi-center adelic versions, on curves, of the other classical applications of capacity theory:

Bilu's Equidistribution Theorem on Curves: A. Thuillier (Thesis, University of Rennes, 2005);

The Dynamical Equidistribution Theorem for Small Points (Baker-Rumely 2006, Favre-Rivera Letelier 2006, Chambert-Loir, Thuillier, and Autissier 2007)

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New Applications

There are also new applications of capacity theory, using extremal extremal properties of Green's functions and equilibrium measures, energy minimization principles, and equidistribution principles, by Igor Pritsker, Paul Fili, Zachary Miner, Clayton Petsche, Lukas Pottmeyer, and others.

Arithmetic Capacity on Higher Dimensional Varieties

Let K be a global field. For varieties V/K of dimension $d \geq 2$, there is a good notion of arithmetic capacity in place:

$$\text{Chinburg's Sectional Capacity } S_\gamma(\mathbb{E}, D).$$

Chinburg (1991) has proved a higher dimensional analog of Fekete's Theorem.

Xinyi Yuan (2008) has given a higher-dimensional analogue of the Equidistribution Theorem for small points.

There is as yet no higher dimensional analogue of the Fekete-Szegö Theorem or the Pol'ya-Carlson Theorem. However, Bost and Chambert-Loir have obtained results related to the Pol'ya-Carlson Theorem, using Arakelov Theory and the Method of Slopes.

Very recently, Chinburg, Moret-Bailly, Pappas, and Taylor found a higher-dimensional analogue of the Cantor Capacity.

v-Basic Sets

If v is archimedean and $K_v \cong \mathbb{C}$, a set $F_v \subset \mathcal{C}(\mathbb{C})$ is v-basic if it is simply connected, has a piecewise smooth boundary, and is the closure of its interior.

If v is archimedean and $K_v \cong \mathbb{R}$, a set $F_v \subset \mathcal{C}(\mathbb{C})$ is v-basic if either

- it is simply connected, has a piecewise smooth boundary, and is the closure of its $\mathbb{C}$-interior; or
- it is contained in $\mathcal{C}(\mathbb{R})$ and is homeomorphic to a segment $[a, b]$.

If v is nonarchimedean, a set $F_v \subset \mathcal{C}(\mathbb{C}_v)$ is v-basic if

- it is an open ball $B(a, r)^-$ or a closed ball $B(a, r)$; or
- it is a closed affinoid in the sense of rigid analysis; or
- for some separable algebraic extension L_w/K_v (finite or infinite), it is the intersection of $\mathcal{C}(L_w)$ with an open or closed ball or an affinoid.

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Example: A disc with a tail

Theorem

Let $0 < R, L \in \mathbb{R}$, and take $E_\infty = D(0,R) \cup [R, R+L]$, a 'disc with a tail'. Fix a prime p, and let

$$E_p = p\mathbb{Z}_p^\times \cup \mathbb{Z}_p^\times \cup p^{-1}\mathbb{Z}_p^\times = \mathbb{Q}_p \cap (D_p(0,p) \backslash D_p(0,1/p)^-) ,$$

a p-adic annulus. For each prime $q \neq p$, put $E_q = D_q(0,1)$. Then if

$$\left(\tfrac{3}{4}R + \tfrac{1}{4}\tfrac{R^2+RL+L^2}{R+L}\right) \cdot p^{1-\frac{1}{p-1}+\frac{1}{(p-1)^2(1+p^2+p^4)}} > 1 ,$$

there are infinitely many algebraic numbers whose whose conjugates in $\mathbb{C}_v$ belong to E_v, for each place v.

If the reverse inequality holds, there are only finitely many.

Example: A disc with a tail

The sets in the example are finite unions of 'basic sets':

The set E_∞ is a union of a set in $\mathbb{C}$ which is the closure of its complex interior, and a set in $\mathbb{R}$ which is the closure of its real interior. Note that these sets need not be disjoint.

The set E_p is a union of affine translates of $\mathbb{Z}_p$:

$$E_p = \bigcup_{i=-1}^{1} \bigcup_{a=1}^{p-1} \left(a \cdot p^i + p^{i+1}\mathbb{Z}_p \right) .$$

The sets $E_q = D_q(0,1)$ for $q \neq p$ are 'trivial' with respect to ∞.

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An Elliptic Curve example

Let $\mathcal{E}/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 - 256x$.

The real locus $\mathcal{E}(\mathbb{R})$ has two components, with a bounded loop lying over the interval $[-16, 0]$.

Theorem

There are infinitely many points $\alpha \in \mathcal{E}(\widetilde{\mathbb{Q}})$ whose archimedean conjugates all belong to the bounded real loop of $\mathcal{E}(\mathbb{R})$, whose 2-adic conjugates all belong to $\mathcal{E}(\mathbb{Z}_2)$, and whose p-adic conjugates all belong to $\mathcal{E}(\widehat{\mathcal{O}}_p)$ where $\widehat{\mathcal{O}}_p$ is the ring of integers of $\mathbb{C}_p$

Here $\mathfrak{X} = \{\overline{o}\}$ (the origin of $\mathcal{E}$), and $\gamma(\mathbb{E}, \mathfrak{X}) = \prod_v \gamma_{\overline{o}}(E_v)$ where

$$\gamma_{\overline{o}}(E_\infty) = 2, \quad \gamma_{\overline{o}}(E_2) = 2^{-106/107}, \text{ and } \quad \gamma_{\overline{o}}(E_p) = 1 \text{ for all odd } p.$$

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Take $K = \mathbb{Q}$ and consider the Fermat Curve $\mathcal{F}$ with affine equation $x^p + y^p = 1$.

It has p points at ∞; let $\mathfrak{X}$ be that set of points.

Take $0 < R \in \mathbb{R}$ and put $E_\infty = \{(x,y) \in \mathcal{F}(\mathbb{C}) : |x| \leq R\}$.
At the prime p, let $L_w/\mathbb{Q}_p$ be the extension $L_w = \mathbb{Q}_p(\zeta_p)$.
Put $E_p = \mathcal{F}(\mathcal{O}_{L_w})$.
For all other primes q, let E_q be the $\mathfrak{X}$-trivial set.

McCallum has determined a regular model for $\mathcal{F}$ over $\mathcal{O}_{L_w}$; it has n_p components of a certain type, corresponding to the number of nontrivial linear $\mathbb{F}_p$-rational factors of the equation $((x-y)^p - (x^p - y^p))/p \equiv 0 \pmod{p}$.

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McCallum has determined a regular model for $\mathcal{F}$ over $\mathcal{O}_{L_w}$; it has n_p components of a certain type, corresponding to the number of nontrivial linear $\mathbb{F}_p$-rational factors of the equation $((x-y)^p - (x^p - y^p))/p \equiv 0 \pmod{p}$.

A Fermat Curve example

Take $K = \mathbb{Q}$ and consider the Fermat Curve $\mathcal{F}$ with affine equation $x^p + y^p = 1$.

It has p points at ∞; let $\mathfrak{X}$ be that set of points.

Take $0 < R \in \mathbb{R}$ and put $E_\infty = \{(x, y) \in \mathcal{F}(\mathbb{C}) : |x| \leq R\}$.
At the prime p, let $L_w/\mathbb{Q}_p$ be the extension $L_w = \mathbb{Q}_p(\zeta_p)$.
Put $E_p = \mathcal{F}(\mathcal{O}_{L_w})$.
For all other primes q, let E_q be the $\mathfrak{X}$-trivial set.

McCallum has determined a regular model for $\mathcal{F}$ over $\mathcal{O}_{L_w}$; it has n_p components of a certain type, corresponding to the number of nontrivial linear $\mathbb{F}_p$-rational factors of the equation $((x - y)^p - (x^p - y^p))/p \equiv 0 \pmod{p}$.

A Fermat Curve example

Theorem

There are infinitely many points of $\mathcal{F}(\widetilde{\mathbb{Q}})$ which have all their conjugates in E_v for each v if

$$R \cdot p^{-\frac{p(2p-1)}{(p-1)^2((2n_p+2)p-n_p)}} > 1,$$

and only finitely many if the opposite inequality holds.