Simple linear relations between conjugate algebraic numbers

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Papers


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Pisot numbers

- **Pisot number:**
  A real algebraic integer $\alpha > 1$ whose algebraic conjugates over $\mathbb{Q}$ $\alpha' \neq \alpha$ satisfy $|\alpha'| < 1$.

- **Example:** Golden ratio

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  \alpha = \frac{1 + \sqrt{5}}{2} = 1.61803\ldots, \quad \alpha' = \frac{1 - \sqrt{5}}{2} = -0.61803\ldots
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  Minimal polynomial:

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Figure: Roots of $f(x) = x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$

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Arithmetical neighbours: Salem numbers

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A. Dubickas and C. J. Smyth investigated the configurations of lines in \( \mathbb{C} \) joining the pairs of conjugates of an algebraic number.

- **Theorem 1**
  No three conjugates of a Salem number \( \alpha \) lie on a line.

- **Theorem 2**
  No two lines that pass through two distinct conjugates of a Salem number are parallel, apart from \( d/2 - 1 \) lines parallel to the imaginary axis passing through complex conjugate pairs.

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Previous results

Lemma 3 (Mignotte, 1984)

The equality $\alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_d^{k_d} = 1$ with algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_d$ that are conjugates of a Pisot number $\alpha$ of degree $d$ over $\mathbb{Q}$ and $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ can only hold if $k_1 = k_2 = \cdots = k_d$.

Corollary 4

No two non-real conjugates of a Pisot number have the same argument.

This follows by applying Mignotte’s result to the multiplicative relation $\overline{\alpha_1} \overline{\alpha_2} \alpha_2^{-1} \alpha_1^{-1} = 1$.

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Let $\mathcal{L}$ be a line that is parallel to the real or imaginary axis. How many conjugates of a Pisot number lay on $\mathcal{L}$?

Figure: Conjugates of Pisot numbers on vertical and horizontal lines

- Note: if $\mathcal{L}$ is the real axis $\Im(z) = 0$, the answer is trivial: it is the number of real conjugates of $\alpha$. 

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Problem 6
Does there exist a Pisot number $\alpha$ with conjugates $\alpha_1, \alpha_2, \alpha_3$, such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \text{or} \quad \alpha_1 = \alpha_2 + \alpha_3$$

holds?

Problem 7
Does there exist a Pisot number $\alpha$ with conjugates $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that satisfy at least one of the equations

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4, \quad \alpha_1 = \alpha_2 + \alpha_3 + \alpha_4,$$

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Connection to the geometrical problem

- **Vertical lines:** If
  \[ \Re(\alpha') = \Re(\alpha''), \quad \alpha'' \neq \overline{\alpha'}, \]
  then \( \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 \) holds for
  \[ \alpha_1 = \alpha', \quad \alpha_2 = \overline{\alpha'}, \quad \alpha_3 = \alpha'', \quad \alpha_4 = \overline{\alpha''}. \]

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Theorem 8

If \( \alpha \) is a Pisot number of degree \( d \geq 4 \) whose four distinct conjugates over \( \mathbb{Q} \) satisfy the relation

\[
\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4
\]

then

\[
\alpha = \frac{1 + \sqrt{3 + 2\sqrt{5}}}{2}.
\]

Moreover, there exists no Pisot number \( \alpha \) whose four distinct conjugates satisfy the linear relation

\[
\pm \alpha_1 = \alpha_2 + \alpha_3 + \alpha_4.
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Main results, part I: four term equations

▶ **Theorem 8**

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The Pisot number given in Theorem 8 is
\[ \alpha = \alpha_1 = (1 + \sqrt{3 + 2\sqrt{5}})/2 = 1.8667 \ldots \]
has a minimal polynomial
\[ f(x) = x^4 - 2x^3 + x - 1. \]
It has another real conjugate
\[ \alpha_2 = (1 - \sqrt{3 + \sqrt{5}})/2, \]
and two non-real conjugates
\[ \alpha_3 = (1 + \sqrt{-3 + 2\sqrt{5}i})/2, \quad \alpha_4 = (1 - \sqrt{-3 + 2\sqrt{5}i})/2. \]
They satisfy the linear relation
\[ \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = 1. \]
Main results, part II: geometrical consequences

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Corollary 9
There exists no Pisot number with two non-real conjugates having the same imaginary part.

Corollary 10
At most two conjugates of a Pisot number can have the same real part, in which case they are complex-conjugate to each other.
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Theorem 11

If \( \alpha \) is a Pisot number having at least three conjugates over \( \mathbb{Q} \) satisfying the relation

\[ \alpha_1 + \alpha_2 + \alpha_3 = 0 \]

then \( \alpha \) is Siegel’s number \( \theta = 1.32471 \ldots \) (the root of \( x^3 - x - 1 = 0 \)). Furthermore, there does not exist a Pisot number \( \alpha \) whose three conjugates satisfy the relation

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$$\alpha_1 = \alpha_2 + \alpha_3.$$
Demonstration for the equation $\pm\alpha_1 = \alpha_2 + \alpha_3$

- We can assume $\alpha = \alpha_1$ is a Pisot number. From $\pm\alpha_1 = \alpha_2 + \alpha_3$ we obtain $\alpha < 2$.

- Lemma 12 (Beukers and Zagier, 1997)

  Let $\beta_1, \ldots, \beta_r$ be non-zero algebraic numbers such that their sum $N = \beta_1 + \cdots + \beta_r$ is a rational integer. If

  $$\beta_1^{-1} + \cdots + \beta_r^{-1} \neq N \quad (1)$$

  then $h(\beta_1) + \cdots + h(\beta_r) \geq \frac{1}{2} \log \left(\frac{1+\sqrt{5}}{2}\right)$. Here, $h(\gamma)$ stands for the Weill height of an algebraic number $\gamma$.

- If $\gamma$ or $-\gamma$ is algebraically conjugate to a Pisot number $\alpha$, then $h(\gamma) = \log \alpha / d$, where $d = \deg(\alpha)$.
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Demonstration continued

Note: Take $\beta_1 = \mp \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$. One easily checks that $\beta_1 + \beta_2 + \beta_3 = 0$ and $\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1} \neq 0$: otherwise, one would have impossible relation $\alpha_1^2 = \alpha_2 \alpha_3$.

Beukers–Zagier inequality gives

$$\frac{3 \log \alpha}{d} = 3h(\alpha) = h(\pm \alpha_1) + h(\alpha_2) + h(\alpha_3) =$$

$$= h(\beta_1) + h(\beta_2) + h(\beta_3) \geq \frac{1}{2} \log \left( \frac{1 + \sqrt{5}}{2} \right),$$

or

$$\frac{3 \log \alpha}{d} \geq \frac{1}{2} \log \left( \frac{1 + \sqrt{5}}{2} \right).$$

Since $1 < \alpha < 2$,

$$d \leq \frac{6 \log 2}{\log \left( \frac{1 + \sqrt{5}}{2} \right)} = 8.64252 \ldots.$$
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  d \leq \frac{6 \log 2}{\log \left( \frac{1 + \sqrt{5}}{2} \right)} \approx 8.64252 \ldots.
  \]
Demonstration continued

► **Note:** Take $\beta_1 = \mp \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$. One easily checks that $\beta_1 + \beta_2 + \beta_3 = 0$ and $\beta_1^{-1} + \beta_2^{-1} + \beta_3^{-1} \neq 0$: otherwise, one would have impossible relation $\alpha_1^2 = \alpha_2 \alpha_3$.

► Beukers–Zagier inequality gives

$$\frac{3 \log \alpha}{d} = 3h(\alpha) = h(\pm \alpha_1) + h(\alpha_2) + h(\alpha_3) =$$

$$= h(\beta_1) + h(\beta_2) + h(\beta_3) \geq \frac{1}{2} \log \left( \frac{1 + \sqrt{5}}{2} \right),$$

or

$$\frac{3 \log \alpha}{d} \geq \frac{1}{2} \log \left( \frac{1 + \sqrt{5}}{2} \right).$$

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$$d \leq \frac{6 \log 2}{\log \left( \frac{1 + \sqrt{5}}{2} \right)} = 8.64252 \ldots$$
Consequently, $d$ can be only $d = 3, 4, 5, \ldots, 8$. Plugging these $d$ back, one gets refined intervals:

$$\left(\frac{1 + \sqrt{5}}{2}\right)^{d/6} \leq \alpha < 2, \quad 3 \leq d \leq 8.$$  

We computed all Pisot numbers in these intervals up to degree 8 by using Boyd’s algorithm. In total, 78 Pisot numbers were found.

After testing, it was found that the only solution was produced by Siegel’s polynomial $x^3 - x - 1 = 0$. 
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Computations for the four term equations

- Similar methods can be used to show that all Pisot numbers whose conjugates satisfy any of the four therm linear relation must lie in the intervals

\[
\left(\frac{1 + \sqrt{5}}{2}\right)^{d/8} \leq \alpha < 3, \quad 4 \leq d \leq 18.
\]

- To find all Pisot numbers in a given interval up to a given degree, we wrote a fast implementation of the Boyd’s algorithm in the C language based on FLINT (Fast Library for Number Theory) version 2.4.4.

- The initial searches for small degrees (up to \(d \leq 17\)) were done on a single RedHat Linux server equipped with two Intel Xeon X5672 series 3.20GHz 12MB Cache 1333MHz 95W CPUs and 96735MB of RAM at the University of Waterloo.
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### Table: Single Intel Xeon 3.4GHz machine search timings

<table>
<thead>
<tr>
<th>deg $\alpha$</th>
<th>Search interval</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>[1, 3]</td>
<td>25 sec.</td>
</tr>
<tr>
<td>11</td>
<td>[1, 3]</td>
<td>2 min. 13 sec.</td>
</tr>
<tr>
<td>12</td>
<td>[1, 3]</td>
<td>11 min. 7</td>
</tr>
<tr>
<td>13</td>
<td>[1, 3]</td>
<td>54 min.</td>
</tr>
<tr>
<td>14</td>
<td>[1, 3]</td>
<td>4 h. 13 min.</td>
</tr>
<tr>
<td>15</td>
<td>[1, 3]</td>
<td>18 h. 47 min.</td>
</tr>
<tr>
<td>16</td>
<td>[610/233, 3]</td>
<td>3 days 11 h.</td>
</tr>
<tr>
<td>17</td>
<td>[367/132, 3]</td>
<td>13 days 17 h.</td>
</tr>
<tr>
<td>18</td>
<td>[437/148, 3]</td>
<td>$\leq 60$ days (estimated)</td>
</tr>
</tbody>
</table>
Consequently, the computations were distributed on a large collection of 2 Intel 5272 series 3.4Ghz/6M/1600Mhz 80W Dual Core Xeon Processor machines, allowing up to 120 simultaneous searches to be done. This was done by partitioning the search interval into 2868 subintervals.

Distributed computations took 13.64 CPU days. In total, 1,955,183 Pisot numbers were found. They are counted in Table 2.
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Computations (continued)

Table: # of Pisot numbers $\alpha \in (\tau^{\deg(\alpha)/8}, 3)$

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<td>4</td>
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<td>14</td>
<td>140 587</td>
</tr>
<tr>
<td>5</td>
<td>162</td>
<td>10</td>
<td>9 937</td>
<td>15</td>
<td>273 851</td>
</tr>
<tr>
<td>6</td>
<td>353</td>
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<td>23 410</td>
<td>16</td>
<td>402 209</td>
</tr>
<tr>
<td>7</td>
<td>1 075</td>
<td>12</td>
<td>40 812</td>
<td>17</td>
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After that, the minimal polynomials of Pisot numbers were tested for possible linear relations among roots (numerically and by using the resultants). There was only one solution detected, namely, a Pisot number with the minimal polynomial $f(x) = x^4 - 2x^3 + x - 1$. 
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More on 3-term equations

Note: Assume that $\alpha$ can be arbitrary algebraic number (not necessarily Pisot!).

Theorem 13
Let $d$ be an integer in the range $3 \leq d \leq 8$ and let $\alpha$ be an algebraic number of degree $d$ over $\mathbb{Q}$. Then some three of its conjugates $\alpha_1, \alpha_2, \alpha_3$ satisfy the relation

$$\alpha_1 = \alpha_2 + \alpha_3$$

if and only if $d = 6$ and the minimal polynomial of $\alpha$ over $\mathbb{Q}$ is an irreducible polynomial of the form

$$f(x) = x^6 + 2ax^4 + a^2x^2 + b \in \mathbb{Q}[x].$$
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$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

if and only if $d = 6$ and the minimal polynomial of $\alpha$ over $\mathbb{Q}$ is an irreducible polynomial of the form

$$f(x) = x^6 + 2ax^4 + 2bx^3 + (a^2 - c^2 t)x^2 + 2(ab - cet)x + b^2 - e^2 t$$

for some rational numbers $a, b, c, e \in \mathbb{Q}$ and some square-free integer $t \in \mathbb{Z}$. 
Basic steps for $\pm\alpha_1 = \alpha_2 + \alpha_3$ up to $d \leq 8$:

- Low degree cases $d = 3$, $d = 4$ and prime degree cases $d = 5$, $d = 7$ are eliminated by elementary considerations and Kurbatov’s lemma.

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- Case $d = 8$, $\alpha_1 = \alpha_2 + \alpha_3$ is most complex: use combinatorics and evaluate group determinants for the Galois group of $\alpha$.

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The End

Thank you!
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