The Geometry, Algebra and Analysis of Algebraic Numbers

Francesco Amoroso (Caen), Igor Pritsker (Oklahoma State),
Christopher Smyth (Edinburgh), Jeffrey Vaaler (UT Austin)

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The participants gathered in Banff on 4 October 2015, smiled upon mild, sunny weather, with clear skies giving almost uninterrupted views of the spectacular surrounding mountains.

They came to the workshop from a variety of mathematical backgrounds: most were number-theorists of various kinds: algebraic, transcendental, analytic, computational, ... The common thread among them was, of course, an interest in algebraic numbers.

1 Overview of the Field and Recent Developments

Algebraic numbers are fundamentally arithmetic objects, a property coming from the integer coefficients of their minimal polynomials. However, their study can involve the deployment of a surprisingly diverse arsenal of mathematical techniques. As well as classical algebra (fields, Galois theory, ...), these include analytic methods [2, 32], algebraic number theory [17, 19], combinatorial methods [29], methods from algebraic and toric geometry [5], potential theory [33, 35], dynamical systems [21, 41]. Classical geometric methods are used, for instance, for studying the position of algebraic numbers in the complex plane [9]. Connections with von Neumann algebras have been revealed, leading to formulations of natural non-commutative analogues of some classical questions about algebraic numbers. Algebraic numbers coming from restricted classes arise naturally in the study of hyperbolic manifolds.

Mahler measure and heights. The workshop capitalized on recent progress and stimulated further development in this area, where the central problem is the 80-year old Lehmer conjecture [26, 42], on the smallest Mahler measure of a nonzero noncyclotomic algebraic integer. Thus, while this fundamental problem is old and difficult, there has been steady progress towards its positive solution in recent years. Some important advances include its proofs for polynomials having a bounded number of monomials [16], with odd coefficients [11] and for algebraic numbers that generate Galois extensions of the rationals [5], generalizations of Dobrowolski-type bounds for multiplicatively independent algebraic numbers [5] and for multivariate polynomials [4], and an absolute lower bound for the height in abelian extensions [6]. The polynomials of smallest Mahler measure coming from integer symmetric matrices have also been found [31]. There are many rapidly developing related directions such as computing explicit values of the Mahler measure, its connection with algebraic geometry and dynamics [41], counting results [28], infinite fields with the Bogomolov [10] and the Northcott property [20], etc.

The legacy of Schur and Siegel. In is of great interest to highlight the inheritance of Schur and Siegel in this area. Their achievements in other areas are so substantial that these have perhaps not received enough attention up to now. One important topic is the Schur-Siegel-Smyth trace problem on the smallest limit point for arithmetic means for totally real and positive algebraic integers [39, 40, 43, 1]. Another is the application of the Schur theory of Hardy functions to the study of Pisot numbers [14], and, more generally, to the theory of nonreciprocal polynomials. The distribution of sets of conjugates algebraic numbers in the complex plane, whose study began with Kronecker [25], and was continued by Schur [39] and Fekete [23], is still far from being well understood. A notable exception is the area of asymptotic equidistribution for algebraic numbers of small height (or small Mahler measure), which received substantial attention in recent years (see, e.g., [8, 44, 22, 45]). The ideas of equidistribution were successfully applied in many problems, including the Schur-Siegel-Smyth trace problem, cf. [35].

Relations between conjugate algebraic numbers. The problem of how conjugate algebraic numbers may be connected algebraically is poorly understood – the difficulty of Lehmer's problem being a consequence of this. A related problem is to find strong lower bounds for discriminants of algebraic numbers, current known exponential lower bounds all coming from consideration of field discriminants. A breakthrough on this would have consequences for Lehmer's problem. Other connections between conjugates have been studied in [7, 17].

Integer Chebyshev problem and other extremal problems for integer polynomials. The problems of minimizing norms by polynomials with integer coefficients date back to at least the work of Hilbert of 1894 [24]. They were developed by Fekete and many others [3, 12, 34], but the integer Chebyshev problem remains open even in its classical setting. It has intimate connections with the distribution of conjugate algebraic integers, and with the Schur-Siegel-Smyth trace problem.

Applications of functional analysis. Recent work has lead to new theorems about the Weil height of algebraic numbers using techniques from functional analysis. In some cases the results can be stated in the classical language of algebraic number theory, but the proofs use a special Schauder basis for the Banach spaces identified in [2]. At present proofs using classical methods of algebraic number theory are not known, but are highly desirable to develop.

2 Open Problems

The workshop incorporated three problem sessions where many participants submitted new open problems and exchanged ideas on the well known ones. A complete collection of discussed open problems is attached to this report due to its size. We state only several central problems below.

1. **Lehmer's problem.** Is there an absolute constant $C > 1$ such that, for any nonzero algebraic integer α, the product of the moduli of those conjugates of α (including α itself) that are of modulus at least 1 is either 1 or at least C.

   The example of α being a zero of "Lehmer's polynomial"

   $$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

   shows that such a C, if it exists, is at most $1.176\ldots$. An optimistic form of this problem, called 'Lehmer's conjecture (strong form)', says that this constant C does exist, and equals $1.176\ldots$.

   For references to Lehmer's problem see for instance [42].

2. **The Schur-Siegel-Smyth trace problem.** For which real numbers r are there only a finite number of totally real and positive algebraic integers β such that $\mathrm{Trace}(\beta)/\deg(\beta)$ is less than r?

   It is known that this is true for all $r < 1.79193$ – see Liang and Wu [27] – but false for $r \geq 2$. An obvious conjecture is that it is true for all $r < 2$, but doubt has been thrown on this by the result of Smyth, improved by Serre (see Appendix to Aguirre and Peral [1]), which shows that the 'auxiliary function' method used to obtain results of this type cannot produce bounds greater than 1.8983021.

   For a survey of the trace problem see [1].

3. **Algebraic integers contained in an interval of length 4.** Pólya proved that any interval $[a, b] \subset \mathbb{R}$ of length $b - a < 4$ contains only finitely many complete sets of conjugate algebraic integers, see [39, p.

391]. If $b-a>4$ then each $[a,b]$ contains infinitely many complete sets of conjugate algebraic integers by the result of Robinson [37]. It is well known that intervals of the form $[a, a+4],\ a\in\mathbb{Z}$, also contain infinitely many such sets. For example, one can consider the roots of polynomials $2T_n(x/2)$ contained in $[-2,2]$ for all $n\in\mathbb{N}$, where $T_n(x)=\cos(n\arccos x)$ is the Chebyshev polynomial.

Problem. Characterize all $a\in\mathbb{R}$ such that each interval $[a, a+4]$ contains infinitely many complete sets of conjugate algebraic integers.

A survey of related problems with further references can be found in McKee [30].

4. **The conjecture of Schinzel and Zassenhaus.** For a nonzero algebraic integer α, of degree d say, $\mathrm{house}(\alpha)$ is the maximum modulus of $|\alpha|$ and any of its conjugates. The Schinzel-Zassenhaus conjecture [38] asserts that there is a constant $c>1$ such that $(\mathrm{house}(\alpha))^d$ is either 1 or at least c.

   If the conjecture is true, there would, for fixed d, be a constant $c_d\ge c$ such that for all α of degree d then $(\mathrm{house}(\alpha))^d$ is either 1 or at least c_d. Boyd [15] has conjectured that for d divisible by 3 the largest value for c_d is $c_d=\theta_0$, the real zero of z^3-z-1, attained for $d=3k$ and α a zero of $z^{3k}+z^{2k}-1$. Computation by Boyd [15] and also by Rhin and Wu [36] provide evidence for this strong conjecture, and also suggest possible values for c_d for other values of d.

   The strongest result to date in the direction of the conjecture is due to Dubickas [18], who proved that

   $$\mathrm{house}(\alpha)\ >1+(\tfrac{64}{\pi^2}-\varepsilon)(\log\log d/\log d)^3/d \quad \text{for } d>d_0(\varepsilon).$$

   The Schinzel-Zassenhaus conjecture is readily seen to be a consequence of 'Lehmer's conjecture' – see [42].

5. **The integer Chebyshev problem.** For a real interval I, its Chebyshev constant given by

   $$t(I)=\inf_P \max_{x\in I}|P(x)|^{1/\deg P},$$

   where the infimum is taken over all non-constant monic polynomials in $\mathbb{C}[x]$. The *integer Chebyshev constant* $t_{\mathbb{Z}}(I)$ has a similar definition, except that now the infimum is taken over all not identically zero (not necessarily monic) polynomials in $\mathbb{Z}[x]$. It is known from the results of Fekete [23] that

   $$\frac{|I|}{4}\le t_{\mathbb{Z}}(I)\le\frac{\sqrt{|I|}}{2},$$

   where $|I|$ is the length of I. It is also known that $t_{\mathbb{Z}}(I)=1$ when $|I|\ge 4$. However, no exact value for $t_{\mathbb{Z}}(I)$ is known for any interval of length between 0 and 4. Can you find $t_{\mathbb{Z}}(I)$ for any interval I with $|I|<4$?

   For more information and references see for instance Amoroso [3], P. Borwein and Erdélyi [12] and Pritsker [34]. There is also a version of the problem where the polynomials over which the infimum is taken *are* restricted to be monic, as well as having integer coefficients – see P. Borwein, Pinner and Pritsker [13].

3 Presentation Highlights

Shabnam Akhtari, University of Oregon

Akhtari is one of a small group of number theorists working on the difficult problems related to the Thue Diophantine equation. Let $F(x,y)$ be an absolutely irreducible, homogeneous polynomial with integer coefficients, and degree greater than or equal to three. A basic problem in number theory is to describe the set of integer solutions to the Diophantine equation $F(x,y)=m$ for nonzero integers m. It is know from the work of Thue that for each m such an equation has only finitely many solutions. Current research is directed at estimating the number of solutions and the height of solutions. Akhtari gave a general overview of such questions, and reported on her recent results that bound the number of solutions to the special Thue equations $ax^n-by^n=c$. This work is closely related to the problem of finding effective measures of irrationality for the algebraic numbers that are roots of the equation $F(x,1)=0$.

Fabrizio Barroero, Scuola Normale Superiore di Pisa

The topic of counting algebraic integers as well as integer polynomials of bounded height is steadily gaining popularity. One of the pioneering results in this direction was obtained by Chern and Vaaler, who proved an asymptotic formula for the number of polynomials with bounded Mahler measure. It is possible to recast their formula as an asymptotic for the number of algebraic integers of bounded degree and height. A generalization of this asymptotic result was proved by Widmer in a multiterm form. Barroero used similar techniques to count monic polynomials by counting lattice points in a more general setting of polynomials over arbitrary number field of fixed degree. This requires to use a recent result of Barroero and Widmer on counting points of an arbitrary lattice in definable sets in an o-minimal structure. The main term of the asymptotic result comes from the volume of a certain bounded domain defined by a generalization of the Mahler measure.

Yann Bugeaud, Université de Strasbourg

In his talk "On the approximation of transcendental numbers by algebraic numbers of bounded degre", Yann Bugeaud has surveyed recent and less recent results closely related to a long-standing conjecture formulated by Wirsing in 1961 on approximation of real numbers by algebraic numbers of bounded degree. This subject is somehow related to Dujella's talk on root separation of integer polynomials.

Emanuel Carneiro, Instituto nacional de Matematica Pure e Aplicada

Let $\xi_1, \xi_2, \ldots, \xi_N$ be distinct real numbers, let $\delta_m \leq \min\{|\xi_m - \xi_n| : n = 1, 2, \ldots, N, \text{ and } n \neq m\}$ for each $m = 1, 2, \ldots, N$, and let $\Delta = \min\{\delta_1, \delta_2, \ldots, \delta_N\} \leq \min\{|\xi_m - x_n| : m \neq n\}$. Then a generalization of Hilbert's inequality proved by H. L. Montgomery and R. C. Vaughan, asserts that

$$\left|\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\xi_m - \xi_n}\right| \leq \pi \Delta^{-1} \sum_{n=1}^{N} |a_n|^2,$$

for all complex numbers $a_1, a_2, \ldots a_N$. Montgomery and Vaughan also proved a weighted version of the form

$$\left|\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\xi_m - \xi_n}\right| \leq \frac{3\pi}{2} \sum_{n=1}^{N} \delta_n^{-1} |a_n|^2.$$

Inequalities of this sort originated in attempts to prove sharp forms of the large sieve inequality. Since their work it has long been a problem to prove the weighted version with the constant $3\pi/2$ replaced by the constant π, which is known to be best possible. Carneiro described new joint work with F. Littmann which provides a new and very original proof of the second inequality, but with the slightly inferior constant 2π. This is of interest because the Carneiro/Littmann method uses a new type of extremal function related–but not seen before–to the theory of Beurling-Selberg extremal functions. Clearly the Carneiro/Littmann extremal function is not quite the right one; hence the inferior constant. But it is significant progress that the weighted Hilbert inequality is now seen to be connected to the theory Beurling-Selberg extremal functions. It is to be hoped that this will lead to a new extremal function providing a new proof with the conjectured constant π.

Stephen Choi, Simon Fraser University

Several problems of analytic number theory arise from the study of Littlewood polynomials. These are polynomials with all coefficients equal to 1 or -1. There name is attached to questions raised by Littlewood concerning the size of such polynomials on the unit circle. By Parseval's identity a Littlewood polynomial $F(x)$ of degree N satisfies

$$\left(\int_0^1 \left|F\left(e^{2\pi i x}\right)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} = \sqrt{N+1}.$$

Littlewood asked if there exist such a polynomial with supremum norm on the circle not much larger than $\sqrt{N+1}$. Stephen Choi is an expert on such questions, and gave a general overview of recent research on bounds for the L^1 norm of Littlewood polynomials.

Paulius Drungilas, Vilnius University

Drungilas discussed a kind of abc problem for algebraic numbers: for which triples (a, b, c) of positive integers do there exist algebraic numbers α, β and γ of degrees a, b, c respectively with $\alpha + \beta + \gamma = 0$? Such triples are called *sum-feasible*. Building on earlier work, he finds, in joint work with Dubickas and Luca, all such triples with $a \le b \le c \le 7$. He also discussed the multiplicative analogue ('product-feasible' triples) where, in the definition, $\alpha + \beta + \gamma = 0$ is replaced by $\alpha\beta\gamma = 1$. He showed with Dubickas that every sum-feasible triple is also product-feasible, but that the two notions are not equivalent since, for instance, the triple $(2, 3, 3)$ is product- but not sum-feasible.

Arturas Dubickas, Vilnius University

In his talk "Counting dominant and degenerate polynomials," Arturas Dubickas presented a wide range of estimates on the number of polynomials with integer coefficients of fixed degree and bounded height that have certain special properties. In particular, he considered dominant polynomials (those that have one root whose modulus is strictly greater than the moduli of the remaining roots) and degenerate polynomials (that have a pair of distinct roots whose quotient is a root of unity). One of the main results shows that asymptotically all monic integer polynomials are dominant, meaning that the proportion of dominant polynomials among all integer polynomials tends to 1 as their heights increase to infinity. If integer polynomials are not monic, then dominant polynomials represent a positive fraction that generally depends on their degree. The asymptotic value of this fraction is known only for some degrees, and in general represents an open problem. The motivation for this study arose from linear recurrence sequences, and all the results are joint with Min Sha (Sydney).

Finally, the speaker stated a series of asymptotic formulas for the number of reducible polynomials and irreducible polynomials in terms of their heights. The latter problem is an old one: in the monic case the corresponding formula was established by Chela in 1963.

Andrej Dujella, University of Zagreb

On the BIRS workshop, Andrej Dujella gave a talk titled "Root separation for reducible integer polynomials", reporting on a recent joint work with Bugeaud. He consider the question how close to each other can be two distinct roots of an integer polynomial $P(X)$ of degree d. He compare the distance between two distinct roots of $P(X)$ with its height $H(P)$, defined as the maximum of the absolute values of its coefficients. The first result in this direction in due to Mahler, who proved that the distance is $> c(d)H(P)^{-d+1}$, for an explicit constant $c(d)$, depending only on d. In the talk, he present some recent results in the opposite direction, obtained by constructing explicit parametric families of (monic) reducible polynomials having two roots very close to each other.

Tamas Erdelyi, Texas A&M University

The order of vanishing for integer polynomials at 1 is an old and well known problem. It was considered for many special classes such as Littlewood, Newman and other kinds of integer polynomials. For example, the difficult problem of Prouhet, Tarry, and Escott in Diophantine equations has an equivalent reformulation as a question of this kind.

Tamas Erdelyi used the Coppersmith-Rivlin type inequalities as his main analytic tool to give estimates for the order of vanishing at 1 for polynomials with restricted coefficients. In fact, he presented various generalizations of the original Coppersmith-Rivlin inequality (obtained in the case of ℓ_∞ norm) to the full range of ℓ_p norms.

Michael Filaseta, University of South Carolina

It is well known that any monic irreducible non-cyclotomic polynomial with integer coefficients must have a root outside the closed unit disk. Many papers are devoted to the estimates on how far from the unit disk such roots must lie, which is directly related to the famous Lehmer conjecture. In this talk, Michael Filaseta discussed analogous regions different from disks for which similar estimates hold. These regions are constructed in a very special way from cyclotomic polynomials.

Michael Filaseta gave a surprising application of such regions to the problem on irreducibility of polynomials with non-negative coefficients. A result of A. Cohn implies that if the coefficients of a polynomial $f(x)$ are non-negative and bounded by 9, and if $f(10)$ is a prime number, then f is irreducible. The new approach using constructed regions allows to replace 9 with the sharp bound 49598666989151226098104244512918 for the coefficients. Moreover, its gives sharp bounds on coefficients under the modified assumption that $f(b)$ is prime for other positive integers b.

Several questions were stated concerning roots in these regions that would lead to further developments in the applications to irreduciblity.

Paul Fili, Oklahoma State University

Zannier asked a question about finiteness of the set of parameters $c \in \mathbb{C}$ such that $z = 0, 1$ are both preperiodic under iteration of $f_c(z) = z^2 + c$. It is now customary to refer to such questions as problems on *unlikely intersections* in arithmetic dynamics. Baker and DeMarco were able to answer this question in the affirmative using equidistribution. Using the metric of mutual energy and some discrete approximation techniques, Paul Fili proved an effective degree bound for the algebraic integers c arising in the question of Zannier. The main idea behind the proof is similar to that of Baker and DeMarco. Namely, the probability measure equally supported on the Galois conjugates of such a number c would tend to the equilibrium measures of both Mandelbroit sets M_0 and M_1 as the degree of c grows, which is impossible as these equilibrium measures are distinct. Hence the degree of c must be bounded. The notion of distance between two (adelic) measures defined via the mutual energy metric, and the triangle inequality for this metric, allow to effectively estimate the distance between the mentioned measures. Together with discrete energy approximation techniques, these ideas give an effective degree bound for the set of parameters c in question.

Robert Grizzard, University of Wisconsin

Robert Grizzard gave a talk entitled "Remarks on diophantine approximation in the multiplicative group and generalized Lehmer problems," discussing the following ideas. The absolute logarithmic Weil height $h : \mathbb{G}_m(\overline{\mathbb{Q}}) \to [0, \infty)$ induces a norm on the $\mathbb{Q}$-vector space $\mathbb{G}_m(\overline{\mathbb{Q}})/\text{tors} = \overline{\mathbb{Q}}^\times \otimes_\mathbb{Z} \mathbb{Q}$ (which is written multiplicatively!), where the torsion subgroup in this case simply consists of roots of unity. If Γ is any subgroup of $\mathbb{G}_m(\overline{\mathbb{Q}})$, we define a height function relative to Γ, written $h_\Gamma : \mathbb{G}_m(\overline{\mathbb{Q}}) \to [0, \infty)$, as follows: $h_\Gamma(\alpha) = \inf_{\gamma\in\Gamma} h(\alpha/\gamma) = \inf_{\gamma\in\Gamma} h(\alpha\gamma)$. If we interpret $h(\alpha/\beta)$ as the distance between α and β, then $h_\Gamma(\alpha)$ is the distance from α to the subgroup Γ. We discuss the result, joint with J. D. Vaaler, that for any field $k \subseteq \overline{\mathbb{Q}}$, we have

$$w_k(\alpha) \geq h_{k^{\text{div}}}(\alpha) \geq \frac{1}{2} w_k(\alpha),$$

where $w_k(\alpha) = \max\left\{h(\sigma\alpha/\alpha) \mid \sigma \in \text{Gal}(\overline{\mathbb{Q}}/k)\right\}$, and $k^{\text{div}} = k^\times \otimes_\mathbb{Z} \mathbb{Q}$.

By combining the lower bound on $h_{k^{\text{div}}}$ with a Dobrowolski-type estimate of Amoroso and Zannier, one achieves a bound of the form

$$h_{k^{\text{div}}}(\alpha) \geq \frac{c}{[k^{ab}(\alpha) : k^{ab}]^{2+\varepsilon}},$$

when k is a number field. Using this, Grizzard explained a connection to the "degree one form" of the generalized Lehmer problems recently proposed by G. Rémond.

Adam Hughes, University of Texas at Austin

Let $\overline{\mathbb{Q}}^\times$ be the multiplicative group of nonzero algebraic numbers, let $\text{Tor}\big(\overline{\mathbb{Q}}^\times\big)$ denote its torsion subgroup, and write $\mathcal{G} = \overline{\mathbb{Q}}^\times/\text{Tor}\big(\overline{\mathbb{Q}}^\times\big)$ for the quotient group. It is known that $\mathcal{G}$ has the structure of a vector space over $\mathbb{Q}$ written multiplicatively. The map $\alpha \mapsto h(\alpha)$ is well defined on $\mathcal{G}$, where $h(\alpha)$ is the absolute logarithmic Weil height of a coset representative, because distinct representatives of each coset differ only by a root of unity. Moreover, elementary properties of the Weil height imply that the map $\alpha \mapsto h(\alpha)$ is a norm on the vector space $\mathcal{G}$ written multiplicatively. Hence the completion of this vector space is a real Banach space. In a recent paper Allcock and Vaaler [2] showed that this Banach space $\mathcal{X}$ is isomorphic to a co-dimension one subspace of $L^1(Y, \mathcal{B}, \lambda)$, where Y is the set of all places of $\overline{\mathbb{Q}}$ with an inverse limit topology, $\mathcal{B}$ is the σ-algebra of Borel subsets of Y, and λ is an invariant measure defined on $\mathcal{B}$ that is induced by the action of the absolute Galois group $\text{Aut}\big(\overline{\mathbb{Q}}/\mathbb{Q}\big)$. The isomorphism is given explicitly by a map $\alpha \mapsto f_\alpha(y)$, where f_α is a continuous function on Y with compact support. If $\mathcal{F} \subseteq \mathcal{X}$ is the collection of all continuous functions on Y with compact support that have the form $y \mapsto f_\alpha(y)$, then $\mathcal{F}$ is dense in $\mathcal{X}$. Hughes gave a report on several recently developed directions for future research in this setting. A basic discovery of Hughes is a Banach algebra structure in the space $L^1(Y, \mathcal{B}, \lambda)$. Possible implications of this discovery for classical problems about the Weil height of algebraic numbers were also described.

Jonas Jankauskas, Vilnius University Jankauskas described joint work with Dubickas where they completely solve the equations $\alpha_1 = \alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$ in conjugate algebraic numbers α_i of degree up to 8. The techniques used are diverse: they involve methods from linear algebra, Galois theory and some combinatorial arguments. They also solve these equations when the α_i are Pisot numbers and their conjugates.

Matilde Lalín, Université de Montréal

Let $F(x_1, x_2, \ldots, x_N)$ be a nonzero Laurent polynomial in N variables with complex coefficients. The Mahler measure of F is given by

$$M(F) = \exp\left\{\int_{(\mathbb{R}/\mathbb{Z})^N} \log\big|F\big(e^{2\pi i x_1}, e^{2\pi i x_2}, \ldots, e^{2\pi i x_N}\big)\big| \,\mathrm{d}\boldsymbol{x}\right\}.$$

If $N = 1$ then Jensen's identity leads to an alternative expression for the value of Mahler's measure as a product over the roots of $F(z) = 0$ which occur outside the unit circle. If $N \geq 2$ no such simple formula is known, and the problem of evaluating $M(F)$ for Laurent polynomials with integer coefficients is much more difficult. It is known, from relatively few examples, to be connected to special values of certain L-functions. More explicit conjectures of Beilinson have guided much recent research in this field. Lalin described recent joint work with D. Samart and W. Zudilin which establishes an identity between the Mahler measure of the Laurent polynomial $x + x^{-1}y + y^{-1} + 3$ and a special value of an associated elliptic curve. This confirms a conjecture of David Boyd based on computer calculations of the two values to very high precision. Several further examples were described.

James McKee and Pavlo Yatsyna, University of London

McKee and Yatsyna discussed a 22-year-old conjecture of Estes and Guralnick to the effect that any monic integer polynomial of degree d without multiple roots will occur as the minimal polynomial of some integer symmetric matrix. This was known to be true only for $d \leq 4$. They show that the conjecture is false for all $d \geq 6$, leaving only $d = 5$ the only unsolved degree. Their idea was to find a lower d-dependent bound for the span of the roots of a minimal polynomial of degree d of an integer symmetric matrix, and then exhibit polynomials that did not satify these bounds, for all sufficiently large d. This disproved the conjecture for those d, and ad hoc methods further eliminated all lower degrees down to $d = 6$. Thus $d = 5$ is the only unsolved case – see problem list.

Michael Mossinghoff, Davidson College

Let $a_1, a_2, \ldots, a_N$, be a finite sequence of integers such that $a_n = \pm 1$ for each $n = 1, 2, \ldots N$. It will be convenient to set $a_n = 0$ if $N + 1 \leq n$. Such a finite sequence is called a *Barker sequence* if it satisfies the condition

$$\sum_{n=1}^{\infty} a_n a_{n+k} \in \{-1, 0, 1\}$$

for every positive integer k. The associated *Barker polynomial* is $P(z) = a_1 + a_2 z + \cdots + a_N z^{N-1}$. From the definition of a Barker sequence it follows that the absolute value of a Barker polynomial on the unit circle is relatively flat. Mossinghoff spoke about the relationship between Barker polynomials and several questions in combinatorics, number theory and analysis. The primary conjecture in this subject asserts that there are only finitely many distinct Barker sequences. Indeed, the longest known Barker sequences has length $N = 13$, and it may be true that no Barker sequence exists with $N > 13$. Mossinghoff described an extensive numerical search for long Barker sequences that produced the following result: if $N > 13$ is the length of a Barker sequence, then either $N = 3979201339721749133016171583224100$, or $N > 4(10)^{33}$. Several additional arithmetic restrictions were described that limit in various ways the possible values of $N > 13$, if N is the length of a Barker sequence.

Lukas Pottmeyer, Universitat Basel

In his very interesting (and beautifully presented) talk "On Narkiewicz's property (P)", Lukas Pottmeyer presented his recent proof of the Narkiewicz conjecture. We say that a field F has property (P) if and only if there is no infinite subset $X \subseteq F$, such that $f(X) = X$ for any polynomial $f \in F[x]$ with $\deg(f) \geq 2$. Let $\mathbb{Q}^{(d)}$ be the compositum of all number fields of degree $\leq d$. In 1963 Narkiewicz conjectured that $\mathbb{Q}^{(d)}$ has property (P) for all positive integer d. He proved his conjecture for $d = 1$. Later on, building on more general results, Bombieri and Zannier (2001) proved the case $d = 2$. Building on a equidistribution theorem for points of small height on the v-adic Berkovich line $\mathbb{P}_v^{\mathrm{Berk}}$ (due to several authors) Pottmeyer proves that a Galois extension F/K with uniformly bounded local degree satisfies a Uniform Bogomolov Property (USB): *any* canonical height associated to a rational map in $F(x)$ of degree > 1 is $> c > 0$ outside the set of zero-height points. This is a deep generalization of one of the main result of Bombieri and Zannier (2001). He then remark that (USB) implies (P).

Igor Pritsker, Oklahoma State University

Pritsker's talk was devoted to the problems of Schur on the limit points for the arithmetic means of conjugate algebraic numbers contained in the closed unit disk and in the real line. There are many recent results on the equidistribution of algebraic numbers of small height according to certain equilibrium measures arising in potential theory. This equidistribution is expressed in terms of weak convergence for the counting measures of algebraic numbers to the corresponding equilibrium measure, so that means of algebraic numbers converge to the first moment (center of mass) of the limiting measure. It is well known that the limiting equilibrium measure for the unit disk is the normalized arclength on the unit circle, and its center of mass is at the origin, which answers a question of Schur about algebraic numbers in the unit disk. Moreover, such methods allow to make more precise quantitative statements about means by using certain discrepancy estimates.

Another application of this approach is the Schur-Siegel-Smyth problem on the smallest limit point of the mean trace of totally positive algebraic integers. It was shown that this smallest limit point is 2 for many classes of algebraic numbers equidistributed in subsets of the real line. This includes algebraic numbers whose minimal polynomials satisfy various extremal properties, as well as those arising in polynomial dynamics. Various generalizations of this problem to the means of powers of algebraic numbers were also presented.

Georges Rhin, Université de Lorraine

Rhin, in collaboration with El Otmani and Sac-Épée, developed an innovative method for finding totally positive algebraic integers of small mean trace. They applied it to find 41 totally positive algebraic integers of degree 17 and trace 31. One of these immediately gives a Salem number of degree 34 and trace -3, the smallest known degree for which a Salem of trace -3 has been found. The idea (e.g., for $d = 17$) is as follows: first find an upper bound (6.69 for $d = 17$) for the largest conjugate. This is done using a particularly effective version of the auxiliary function method due to Flammang. Then choose 16 numbers uniformly at random from $(0, 6.69)$, and search for monic integer polynomials $x^{17} - 31x^{16} + \dots$ that alternate in sign at these values. This is an integer programing problem which, if it is successful in finding such a polyomial that is also irreducible, produces the required algebraic integers. The method is repeated again and again with new random numbers until no new polynomials have been found for some time, making it likely that all polynomials sought have been found.

Robert Rumely, University of Georgia

This talk contained a rather comprehensive survey on the current state of knowledge regarding arithmetic applications of potential theory. This includes the Fekete theorem stating that any compact set of capacity less than one, stable under complex conjugation, contains only finitely many complete sets of conjugate algebraic integers. Another important result is the Fekete-Szegő theorem on existence of infinitely many complete sets of conjugate algebraic integers in every open neighbourhood of a stable compact set of capacity at least one. Robert Rumely also covered the Pólya-Carlson rationality criterion, Ferguson's theorem on functions taking algebraic integer values on a set, and results on approximation of functions by polynomials with integer coefficients. The second part of the talk was devoted to the far reaching generalizations of all of the above classical results, predominantly obtained by the speaker.

Charles Samuels, Christopher Newport University

Samuels described a surprisingly explicit method of computing the Metric Mahler measure of rational numbers p^a/q^b using the continued fraction expansion of $\log q/\log p$. The Metric Mahler measure, being an infimum evaluated over a potentially infinite set, is not easy to compute. Further, in joint work with Jankauskas, Samuels showed that the potentially infinite set can be replaced by a finite one. This set is, however, not easy to find explicitly. For these particular numbers it could in fact be found, and so the Metric Mahler Measure evaluated.

Andrzej Schinzel, Polish Academy of Sciences

By ternary linear recurrence over a number field K we mean a sequence u_n in K satisfying $u_n = a_1u_{n-1} + a_2u_{n-2} + a_3u_{n-3}$, where $a_1, a_2, a_3 \in K$. In his talk, Andrzej Schinzel explained a local-global principle connecting solubility of congruence $u_n \equiv 0 \pmod{P}$ for almost all prime ideals P of K and solubility of the equation $u_n = 0$.

More precisely, Schinzel showed that for every algebraic number field K and every ternary simple linear recurrence u_n in K with the companion polynomial $(z-1)(z-a_1)(z-a_2)$ where $a_1^2 = a_2^x$ for $x = 0, 1$ or 2, if the congruence $u_n \equiv 0 \pmod{P}$ is soluble for almost all, in the sense of density, prime ideals P of K, then the equation $u_n = 0$ is soluble in integers n. A similar result holds if $K = \mathbb{Q}$ and the companion polynomial satisfies $a_1^3 = a_2^x$ for $x = 0, 1$ or 2. In the opposite direction, Schinzel showed that there exist a real quadratic field K and a ternary simple linear recurrence u_n in K with the companion polynomial $(z-1)(z-a_1)(z-a_1^3)$ such that the congruence $u_n \equiv 0 \pmod{P}$ is soluble for all prime ideals P of K, but the equation $u_n = 0$ is not soluble in integers n. The open question is, whether every ternary simple linear recurrence in $\mathbb{Q}$ satisfies the local-global principle.

Christopher Sinclair, University of Oregon

The statistics of roots of a degree N polynomial chosen uniformly from the set of polynomials with Mahler measure bounded by 1 can be analyzed using machinery developed to study the eigenvalues of random matrices. The underlying reason for this is that the joint density of roots contains a Vandermonde term which arises from the Jacobian of the change of variables from roots to coefficients.

The line of research discussed in this talk started with the work of Chern and Vaaler, who computed the volume of the set of (coefficient vectors) degree N polynomials with Mahler measure less than or equal to 1 for both real and complex coefficients. This volume is the main term for an asymptotic estimate on the number of integer polynomials with bounded height. Sinclair later realized that these volume could be interpreted as either the Pfaffian (in the real case) or a determinant (in the complex case) of a Gram-like matrix. The similarity of the partition function calculation in Gaussian ensembles with the Pfaffian volume calculation of Chern and Vaaler, lead Sinclair and Yattselev to introduce the Mahler ensembles: random (real or complex) degree N polynomials chosen uniformly from the set with Mahler measure bounded by 1. Using random matrix machinery, Sinclair and Yattselev recently derived the scaled kernels that give detailed information about the root statistics of random polynomials from the bounded Mahler measure ensemble. They also produced delicate results on the limiting distribution densities of these roots in various sets.

Martin Widmer, University of London

A subset of the algebraic numbers is said to have the Nortcott Property (N) if all of its subsets of bounded height are finite. It is known since 1949 that all number fields have Property (N) but the first examples of infinite degree were given by Bombieri and Zannier in 2001. In particular, they showed that $\mathbb{Q}^{(2)}$, the composite field of all quadratic extensions, has Property (N). However, for $\mathbb{Q}^{(3)}$ there is not much evidence in neither direction. In his talk "Around the Property (N)" Widmer proposed a strategy to make some progress on this problem, by introducing a notion of size for composite fields of cubic extensions. The size is a real number between 0 and 1, and 1 is attained, e.g., for $\mathbb{Q}^{(3)}$. The work of Bombieri and Zannier shows the existence of a subfield of $\mathbb{Q}^{(3)}$ with Property (N) and size 1/2. Widmer uses a different approach (relying on his previous work) that establishes such a subfield with the slightly larger size 3/5.

Qiang Wu, Southwest University of China

Wu made an new contribution to the Schur-Siegel-Smyth trace problem by applying a variant of the auxiliary function method to find the four totally positive monic irreducible reciprocal integer polynomials of smallest mean trace. He used another variant of the method to find the reciprocal algebraic integer of degree d having smallest maximum modulus of its conjugates for every even d up to $d = 42$.

4 Scientific Progress Made

This meeting used an excellent opportunity to bring experts in many different areas together, and enabled them to learn from and build on each others' specialized knowledge. Thus, on the one hand the topic of the workshop was clearly focused, while on the other hand the diversity of participants' interests gave the workshop great potential for cross-fertilisation between different areas of mathematics.

Our workshop was the first event that Pottmeyer participated in, and was focused exactly on his mathematical interests. As a postdoc it is extremely important to make ones research public and to get in contact to many experts in ones field. He enjoyed every talk and could gather new input for further research projects. For instance, he benefited very much from a discussion with Fili and Hughes on the use of equidistribution theorems on the topic of unlikely intersections, a discussion with Widmer on fields with the so called Northcott property, and from a conversation with Schinzel. Some very interesting questions raised by the latter will surely occupy Pottmeyer thoughts for some time.

During the workshop Ranieri and Widmer had several discussion on the connection between Property (N) (a topic that Widmer have studied for a long time) and decidability problems in mathematical logic. This opens a new door and led to interesting new applications of some of Widmer's results and to interesting new questions that might be tackled by number theoretic techniques.

Grizzard was able to work on joint research projects with Vaaler and with Mossinghoff during the workshop. He also found the problem sessions to be very useful for thinking of new research ideas.

Amoroso had a discussion with Smyth after a problem session and was able to make a significant breakthrough towards accurately describing the set $\{h(\alpha)\}$ of Weil heights for $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois.

Carniero and Vaaler had several discussions on the interesting new extremal function that appeared in Carniero's lecture (describing joint work of Carniero and F. Littmann) on the weighted Hilbert inequality. They hope to continue their effort to establish the best constant in the weighted Hilbert inequality by using more refined functions from the general theory of Beurling-Selberg extremal functions.

Dubickas discussed with Pritsker several results related to the problems of Schur on algebraic numbers in various sets. In particular, Dubickas mentioned his earlier results on trace and heights of algebraic integers that are closely related to the questions presented by Pritsker at a problem session. Their communication continued via e-mail after the conference.

Other participants enjoyed very much the inspirational atmosphere of the workshop and interesting discussions with colleagues. Some of them let us explicitly know that they came to the most of the talks, listening with attention and interest to all the speakers.

5 Outcome of the Meeting

The aim of our conference was to preserve and enhance the positive momentum, and to create a strong foundation for further progress in this rich area. We believe that active discussions and emerging collaborations should lead to solutions of important problems. Another significant outcome of the meeting is the list of problems that will be circulated among the participants. The list of problems includes approximately thirty important directions for future research. These were discussed by participants during the problem sessions, and the problems have been somewhat expanded in the written versions. Besides the obviously useful conversations among participants, the extensive list of problems is tangible evidence of a successful meeting.

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Appendix to Report on BIRS workshop 15w5054 on The Geometry, Algebra and Analysis of Algebraic Numbers: Problems proposed by participants

On Mahler's Measure in several variables

(1) (Proposed by Francesco Amoroso).

Let $P \in \mathbb{C}[\mathbf{x}]$ be a polynomial with complex coefficients in the variables $\mathbf{x} = (x_1, \ldots, x_n)$. Its *Mahler measure* is

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log |P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right)| \, dt_1 \ldots dt_n \, . \tag{1}$$

By Kronecker's theorem $M(f) = 1$ for an irreducible polynomial $f \in \mathbb{Z}[x]$ in one variable if and only if $f \neq \pm x$ or if $\pm f$ is a cyclotomic polynomial. Lehmer's problem ask for the existence of an absolute constant $C > 1$ such that $M(f) \geq C$ for any nonconstant irreducible polynomial $f \in \mathbb{Z}[x]$ such that $f \neq \pm x$ and $\pm f$ is not a cyclotomic polynomial.

An irreducible $f \in \mathbb{Z}[\mathbf{x}]$ is an *extended cyclotomic polynomial* if there exist a cyclotomic polynomial ϕ and $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathbb{Z}^n$ such that $f(\mathbf{x}) = \pm \mathbf{x}^{\boldsymbol{\lambda}} \phi(\mathbf{x}^{\boldsymbol{\mu}})$. Let $f \in \mathbb{Z}[\mathbf{x}]$ be irreducible. Then $M(f) = 1$ if and only if $f = \pm x_j$ or if f is an extended cyclotomic polynomial ([2], [4] and [8] independently).

For $P \in \mathbb{C}[\mathbf{x}]$, define the dimension $\dim P$ as the dimension in $\mathbb{R}^n$ of the convex-hull of the set $\{\boldsymbol{\lambda}$ such that $P_{\boldsymbol{\lambda}} \neq 0\}$. It is easy to see that $\dim P$ is the smallest integer m such that f comes from a polynomial in m variables by a monomial transformation:

$$P(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\lambda}_0} Q(\mathbf{x}^{\boldsymbol{\lambda}_1}, \ldots, \mathbf{x}^{\boldsymbol{\lambda}_m}) \tag{2}$$

whith $Q \in \mathbb{C}[y_1, \ldots, y_m]$ and $\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_m \in \mathbb{Z}^n$ (note that this formula implies the equality $M(P) = M(Q)$.) Remark that an irreducible polynomial $f \in \mathbb{Z}[x]$ of dimension $n > 1$ has Mahler's measure > 1, by the quoted result of [2], [4] and [8].

In [3], Boyd asked whether the function

$$m(n) = \inf\{M(f) \text{ such that } f \in \mathbb{Z}[\mathbf{x}] \text{ is irreducible and } \dim f = n\}$$

tends to infinity with n. Very little is known in the direction of Boyd's conjecture. In [1] it is proved that for any irreducible $f \in \mathbb{Z}[\mathbf{x}]$ of dimension $n \geq 2$ we have $M(f) \geq 1/23$ *provided that* the maximum D of its partial degrees is $\leq 3^{2^n}$. On the other hand the best (i.e., of smallest Mahler's measure) known sequence of polynomials seems to be the simplest one: $f_n(x) = 1 + x_1 + \cdots + x_n$ for which we have (see [7], Theorem 3(corrected))

$$M(f_n) = c\sqrt{n} + O(\log n/\sqrt{n}), \quad \text{with} \quad c = e^{-\gamma/2}.$$

Here γ is Euler's constant. For small values of n we can do better. For instance, let $n = 2$ and take any non-reciprocal polynomial $P(x)$ in one variable. Then, letting

$P^*(x) = x^{\deg P} P(1/x)$ be the reciprocal polynomial of P, it is easily see that

$$M(P(x)y - P^*(x)) = M(P) .$$

If P/P^* is not constant, then $P(x)y - P^*(x)$ is irreducible and of dimension 2. Taking for $P(x)$ the minimal polynomial $x^3 - x - 1$ of the smallest Pisot's number $\theta = 1.3247.....$ we get the 2-dimensional irreducible polynomial

$$g(x, y) = x^3 y + x^3 - xy + x^2 - y - 1$$

which has measure $M(g) = \theta < M(1 + x + y) = \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3)\right) = 1.3813...$ (see again [7], example 5).

Problem. Find a family $g_n \in \mathbb{Z}[x_1, \ldots, n]$ of irreducible polynomials of dimension $n \geq 2$ such that $M(g_n) < M(1 + x_1 + \cdots + x_n)$.

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(2) (David Boyd)

(a) Let L be the set of all Mahler measures (defined in (1) above) of polynomials in $\mathbb{Z}[\mathbf{x}]$. Prove that L is a closed subset of $\mathbb{R}$.

(b) Let L_0 be the set of all Mahler measures of nonreciprocal polynomials in $\mathbb{Z}[\mathbf{x}]$. Prove that L_0 is a closed subset of $\mathbb{R}$.

See [1] for details. The second problem may be more tractable than the first.

(c) Does L_0 contain any Salem numbers?

On the other hand, because $M(x^2-qx+1)=M(x^4-qx^3+(q+2)x^2-2x+1)$ for $q\geq 3$ (see [2]), it follows that L_0 contains all Pisot numbers. Incidentally, it also shows that L_0 and the set of Mahler measures of reciprocal polynomials are not disjoint.

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Limit points of Pisot numbers

(3) (David Boyd)

(a) Let $n\geq 1$. Give an algorithm for finding the points of the nth derived set $S^{(n)}$ of the set S of Pisot numbers in a real interval containing no elements of $S^{(n+1)}$.

This would generalize Boyd's algorithm for finding all elements of S in a limit-point-free interval. In this direction Amara [1] determined all elements of $S'(=S^{(1)})$ in the interval $[1,2]$, using Pisot and Dufresnoy′s characterization of S in terms of rational functions $A/Q<1$ on $|z|=1$ except at a finite set [4].

(b) Let $n\geq 1$. Give an algorithm for finding the elements of $S^{(n-1)}$ in a sufficiently small neighbourhood of an element of $S^{(n)}$.

(c) For the set T of Salem numbers, show that its derived set T' is S.

Salem [5, p. 31] wrote 'We do not know whether numbers of T have limit points other than S'. See also Boyd [3, p. 327].

For a comprehensive discussion of Pisot and Salem numbers see [2].

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Graph Pisot and Salem numbers

For the definitions of graph Pisot and graph Salem numbers, see [2].

(4) (a) (Jonas Jankauskas) Can one find all graph Pisot numbers in an interval (say a limit-point free interval, or working up to some degree bound)? This ties in with (6a) and (6b).
(b) Does there exist a totally real Pisot number that that is not a graph-Pisot number? Also, are there any known examples of totally real Pisot numbers that do not appear in the spectra of integer symmetric matrices?
(c) Are there any irrational graph-Pisot numbers, greater than 2, that are known to be in S' or even S'' (first and second derived sets of the set S of all Pisot numbers)?
Of particular interest it would be to locate all such examples inside the open interval $(2, 3)$ (if any exists), or to construct infinite families of such graph-Pisot numbers in S' and especially S'' on the real line.

(5) (James McKee) Can one find all graph Salem numbers in an interval (either limit-point free, or working up to some degree bound)?
The most naïve idea of just looking at all Salem graphs on a given number of vertices doesn't work since there can be cyclotomic factors.

(6) (a) (Chris Smyth) Are all Pisot numbers in a sufficiently small neighbourhood of 2 graph Pisot numbers?
Evidence for this, or a counterexample, could perhaps be obtained by using Boyd's algorithm for finding all Pisot numbers in an interval containing no elements of S'. Now any neighbourhood of 2 contains infinitely many elements of S', so one must use a modified version of the algorithm where you find only elements of S up to a specified degree bound.
(b) The Schur-Dufresnoy-Pisot-Boyd methods of producing Pisot numbers (see e.g., [1]) and graph-based methods [2] seem completely different. Establish a connection between them.

(c) Extend the theory of graph Pisot numbers and graph Salem numbers to signed graph Pisot numbers and signed graph Salem numbers.

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A transcendental number badly approximable by algebraic numbers.

(7) (Yann Bugeaud) Let $n \geq 2$ be an integer. To find an explicit transcendental number ξ such that, for some $c > 0$, we have $|\xi - \alpha| > cH(\alpha)^{-n-1}$ for every algebraic number α of degree at most equal to n. Here $H(\alpha)$ denotes the naïve height of α.

Existence of such numbers was proved only recently by Badziahin and Velani ($n = 2$) and by Beresnevich ($n \geq 2$). I ask for an explicit example of a real number with that property.

A d-variable polynomial problem.

(8) (Tamas Erdélyi)

Lemma Let P_d be a polynomial of exactly d variables with integer coefficients (the degree is irrelevant). Then the maximum modulus of P_d on the d-cube $I_d(2) := [-2, 2]^d$ is at least $(2d)^{1/2}$.

The problem below is also from [1].

Problem As the map $x_1 + x_2 + \cdots + x_d$ suggests, the right lower bound in the Lemma may be $2d$ (or cd). In any case the optimal bound in Lemma 2 is somewhere between $(2d)^{1/2}$ and $2d$. Close the gap. Can the magnitude of the lower bound $(2d)^{1/2}$ in Lemma 1 be improved? Also, are there polynomials P_d of exactly d variables with integer coefficients (the degree is irrelevant) so that the maximum modulus of P_d on the d-cube $I_d(2) := [-2, 2]^d$ is significantly lower than $2d$?

It is not difficult to see that the sharp lower bound is $2d$ in the cases $d = 1$ and $d = 2$.

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Moments of certain dynamical distributions and the Schur-Siegel-Smyth trace problem

(9) (Paul Fili) Let $\mathbb{Q}^{\mathrm{tr}}$ denote the field of all totally real algebraic numbers and $\mathbb{Z}^{\mathrm{tr}}$ its ring of integers. The Schur-Siegel-Smyth trace problem (see Problem 2 of main report) can also be stated as the conjecture that

$$\liminf_{\alpha \in \mathbb{Z}^{\mathrm{tr}}} \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 \geq 2,$$

where above α has degree $[\mathbb{Q}(\alpha) : \mathbb{Q}] = n$ and Galois conjugates $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. This is clearly the smallest possible value, as it is well-known that the value of 2 is attained in the limit by the periodic points of the Chebyshev map $T_2(z) = z^2 - 2$. C.J. Smyth, in an effort to find the smallest possible limiting point of the absolute Weil height for totally real algebraic integers [8], studied the dynamical measure associated to the preperiodic points of the map $H(x) = x - 1/x$. Specifically, he considered the preimages of 1 under the map H:

$$\beta_0 = 1, \quad \beta_n = H^{-1}(\beta_{n-1}) \quad \text{for} \quad n \geq 1,$$

where we choose the positive root for the inverse image. Smyth proved that the β_n are totally real and irreducible algebraic integers, and proved that the logarithmic

Weil height of β_n has a limiting value of $0.27328\ldots$ by studying the distribution of the conjugates of β_n as $n \to \infty$.

Later, Davie and Smyth [2] proved that the numbers β_n^2 also possess the smallest known limit point of the absolute trace of totally positive algebraic integers, and thus are conjecturally minimal for the Schur-Siegel-Smyth trace problem as well. In particular, they proved that:

Theorem. Let β_n be defined as above, and denote by $\beta_n^{(1)}, \ldots \beta_n^{(2^n)}$ its Galois conjugates. Then

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{i=1}^{2^n} \left(\beta_n^{(i)}\right)^2 = 2.$$

In dynamical systems, one can associate to a rational function H of degree $d \geq 2$ a unique canonical Borel probability measure μ_H on the projective line over $\mathbb{C}$ which has no point masses and satisfies the property that $H^*(\mu_H) = d \cdot \mu_H$. As a consequence of the dynamical equidistribution theorem (for the complex case see [7, 4, 5]; for a generalization to non-archimedean fields, see e.g., [1, 3]), μ_H for the map above is the weak limit of the probability measures supported equally on the Galois conjugates of each β_n. It follows that μ_H is supported on the real line. (In fact, Ininger [6, Theorem 4.4] classified the rational maps that have Julia set (equivalent, the support of their canonical measures μ_H) equal to the entire real line.) It follows from the portmanteau theorem on weak convergence of measures that

$$\int x^2\, d\mu_H(x) \geq \lim_{n\to\infty} \frac{1}{2^n} \sum_{i=1}^{2^n} \left(\beta_n^{(i)}\right)^2 = 2.$$

It is easy to show that in fact this is an equality. Consider that for the pullback measure,

$$\int f(x)\, dH^*(\mu_H) = \int H_*(f)(x)\, d\mu_H = 2 \cdot \int f(x)\, d\mu_H$$

for each non-negative Borel measurable f, where the pushforward under H of f is defined to be:

$$H_*(f)(x) = \sum_{y \text{ s.t. } H(y)=x} m_H(y) \cdot f(y),$$

and $m_H(y)$ gives the multiplicity of y in the case of a repeated root. Though x is not nonnegative, provided that $x \in L^1(\mu_H)$, which follows from the finiteness of $\int x^2\, d\mu_H$ below, we see that

$$2\int x\, d\mu_H(x) = \int \left(\frac{x + \sqrt{x^2+4}}{2} + \frac{x - \sqrt{x^2+4}}{2}\right) d\mu_H(x) = \int x\, d\mu_H(x)$$

so

$$\int x \, d\mu_H(x) = 0.$$

For the second moment, we can directly use the pullback of the measure equation to compute

$$\begin{aligned} 2\int x^2 \, d\mu_H(x) &= \int \left(\left(\frac{x+\sqrt{x^2+4}}{2} \right)^2 + \left(\frac{x-\sqrt{x^2+4}}{2} \right)^2 \right) d\mu_H(x) \\ &= \int (2+x^2) \, d\mu_H(x) \end{aligned}$$

so

$$\int x^2 \, d\mu_H(x) = 2.$$

It seems that the key assumptions here are that the rational map had *integer* preperiodic points (or periodic points), and a totally real Julia set. We therefore pose the problem:

Question. Suppose that $F(z) \in \mathbb{Q}(z)$ is a rational map with totally real Julia set, or equivalently by [6, Theorem 4.4], that

(a) $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is totally invariant under F.
(b) There is no attracting fixed point of F in $\mathbb{P}^1(\mathbb{R})$.
(c) Each rationally indifferent fixed point x of F in $\mathbb{P}^1(\mathbb{R})$ is repelling from both sides along the real axis.

Suppose further that $F(z)$ has an infinite number of preperiodic points which are algebraic integers (this is guaranteed if, for example, $F(z) = P(z)/Q(z)$ with $P, Q \in \mathbb{Z}[x]$ with no common factors, $\deg P \neq \deg Q$, and the larger degree polynomial being monic). Let μ_F denote the canonical measure of F. Does the second moment of μ_F always satisfy

$$\int_{\mathbb{R}} x^2 \, d\mu_F(x) \geq 2?$$

I am aware of no such rational map that has smaller second moment, and, as noted above, a positive answer to this question would follow from the conjecture of Schur-Siegel-Smyth.

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Degree problems for various special classes of polynomials

(10) (James McKee)

(a) Is there are degree 5, monic, separable, totally real $f(x) \in \mathbb{Z}[x]$ such that f is not the minimal polynomial of an integer symmetric matrix?
(Change 5 to any other positive integer and the answer is known.)
Some possible candidates: $x^5 - 8x^3 - 5x^2 + 8x + 3$, $x^5 - 7x^3 - 3x^2 + 8x + 3$, $x^5 - 7x^3 - 3x^2 + 7x + 1$.

(b) What is the smallest degree of a Salem number of trace -3? Is it 34?

(c) What is the smallest degree of a Pisot number of trace < 0? Is it 16?

(d) What is the smallest n such that there is a totally real, irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree n with (polynomial) discriminant $< n^n$? Is it 2880?

Newman Representatives of Polynomials with Small Mahler's Measure

(11) (Michael Mossinghoff) At one problem session at the workshop, Jeff Vaaler mentioned the result that every polynomial with integer coefficients having Mahler's measure less than 2 must divide a polynomial with height 1, i.e., having all coefficients in $\{-1, 0, 1\}$. I'd like to ask if a similar, but more specialized, statement may be true if we replace the "height 1" requirement with a stronger restriction: what if we demand all coefficients in $\{0, 1\}$? The set of polynomials with $\{0, 1\}$ coefficients is often called the set of *Newman polynomials*. Clearly, we need to qualify the statement a bit more—a polynomial having a positive real root certainly cannot divide a nonzero Newman polynomial—so let's restrict to polynomials that have no positive real roots. We also need to restrict the measure bound. Recall that every root $\beta \neq 0$ of a height 1 polynomial satisfies $1/2 < |\beta| < 2$, so 2 is a natural bound in the height 1 problem. It is known (Odlyzko & Poonen, *Enseign. Math.* (2) **39** (1993), 317–348) that every root $\beta \neq 0$ of a Newman polynomial satisfies

$1/g < |\beta| < g$, where g is the golden ratio, so one might pick this value, but it turns out that this is asking too much.

Before refining the question, let me mention some evidence toward a possible result in this direction. Artūras Dubickas (*Fiz. Mat. Fak. Moksl. Semin. Darb.* **6** (2003), 25–28) proved that any product of cyclotomic polynomials that has no powers of $z - 1$ must divide a Newman polynomial, and a recent paper of mine with Kevin Hare in *Rocky Mountain J. Math.* **44** (2014), 113–138, shows that the golden ratio is a natural boundary for some other classes of polynomials. We investigated negative Pisot and Salem numbers in $(-g, -1)$: there are four infinite families of negative Pisot numbers in this range (all with limiting value $-g$), plus one sporadic example. All of these negative Pisot numbers which do not have a positive real conjugate divide Newman polynomials. In addition, one can consider the negative Salem numbers one may create by using Salem's construction on all of these negative Pisot numbers. This makes eight doubly-infinite families (from the four families of negative Pisot numbers) plus two singly-infinite families (from the single sporadic example). We found that as long as the negative Salem numbers satisfied the necessary condition of lying in $(-g, -1)$, then it could be represented by a Newman polynomial. Moreover, all 502 negative Salem numbers in $(-g, -1)$ with degree at most 20 can be represented with a Newman polynomial (only about 55% of these are easily generated by Salem's construction on those negative Pisot numbers). Many qualifying polynomials with especially small measure are known to work as well, and some known small limit points of Mahler's measure can be realized by using Newman polynomials.

On the other hand, we know that the measure bound in this statement has to be smaller than g: in the same paper with K. Hare we show for example that $z^6 - z^5 - z^3 + z^2 + 1$ (with measure ≈ 1.556) does not occur as a factor of any Newman polynomial, despite having no roots in disallowed areas. We can now state the question: is there a constant σ so that if $M(f) < \sigma$ and f has no positive real roots then f must divide a Newman polynomial? Assume that f is irreducible if you like. We know $\sigma < 1.556\ldots$ if it exists. Can we take say $\sigma = \sqrt{2}$? Note that the arguments in the corresponding height 1 problem (e.g., Pathiaux (1973), Bombieri & Vaaler (1987)) seem to use the symmetry of the set $\{-1, 0, 1\}$, so this may require a new approach.

Height bounds on the number of irreducible non-cyclotomic factors of a polynomial

(12) (Chris Pinner) For a polynomial $F = \sum_{i=0}^{n} a_i x^i$ in $\mathbb{Z}[x]$ let $H(F) = \left(\sum_{i=0}^{n} a_i^2\right)^{\frac{1}{2}}$. Factoring F in $\mathbb{Z}[x]$ define

$$\Omega_1(F) = \#\text{ irreducible non-cyclotomic factors of } F \text{ counted with multiplicity},$$

$$\omega_1(F) = \#\text{distinct irreducible non-cyclotomic factors of } F,$$

$$m_1(F) = \text{ maximum multiplicity of a non-cyclotomic factor of } F.$$

The truth of Lehmer's Problem straightforwadly leads to the bounds

$$\Omega_1(F) \ll \log H(F), \quad \omega_1(F) \ll \log H(F), \quad m_1(F) \ll \log H(F).$$

Using Siegel's Lemma bounds of this strength are in fact equivalent to Lehmer's Problem.

What bounds can one obtain on $\Omega_1(F)$, $\omega_1(F)$ and $m_1(F)$ that depend solely on $H(F)$?

For example, linear algebra gives

$$m_1(F) < \#\text{ non-zero coefficients of } F \leq H(F).$$

Can this be improved? Is there a similar bound for $\omega_1(F)$ or $\Omega_1(F)$?

The Lind-Mahler Measure and Lind-Lehmer Problem

(13) (Chris Pinner) For a polynomial $F(x_1, \ldots, x_k)$ in $\mathbb{Z}[x_1, \ldots, x_k]$ recall one defines the regular Mahler measure $M(F)$ and logarithmic Mahler measure $m(F)$ by

$$m(F) = \log M(F) := \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi i x_1}, \ldots, e^{2\pi i x_k})| dx_1 \cdots dx_k.$$

Doug Lind (Proc. AMS 2005) viewed $[0,1)^k$ as the group $(\mathbb{R}/\mathbb{Z})^k$ and generalised the Mahler measure to a compact abelian group G with a suitably normalised Haar measure. For example for a finite abelian group

$$G = \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_k}$$

the Lind-Mahler measure $M_G(F)$ and logarithmic Lind-Mahler measure $m_G(F)$ become

$$m_G(F) = \frac{1}{|G|} \log M_G(F) := \frac{1}{|G|} \sum_{x_1=1}^{n_1} \cdots \sum_{x_k=1}^{n_k} \log |F(e^{2\pi i x_1/n_1}, \ldots, e^{2\pi i x_k/n_k})|.$$

The $M_G(F)$ are integers and the polynomials of measure$M_G(F) = 0$ or 1 are the zero divisors or invertible elements in $\mathbb{Z}[x_1, \ldots, x_k]/\langle x_1^{n_1} - 1, \ldots, x_k^{n_k} - 1\rangle$ respectively. Analogous to the Lehmer problem for the regular Mahler measure one can define a Lind-Lehmer constant

$$\lambda(G) = \min\{M_G(F) \geq 2 \ : \ F \in \mathbb{Z}[x_1, \ldots, x_k]\}.$$

(a) **Cyclic Groups:** Now $\lambda(\mathbb{Z}_n)$ is known for $n \neq 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23\, m$. Does $\lambda(\mathbb{Z}_{892371480}) = 25$ or 27?

(b) **p-Groups:** It is known that $\lambda(\mathbb{Z}_p^k) = B_k(p)$ where

$$\begin{aligned} B_n(p) :=& \text{smallest } (p-1)\text{st root of unity mod } p^n \text{ greater than } 1 \\ =& \min\{a^{p^{n-1}} \text{ mod } p^n \ : \ 2 \leq a \leq p-1\}. \end{aligned}$$

What about other p-groups? What is $\lambda(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$?

Note: It can be shown that $B_2(p) \le \lambda(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) \le B_3(p)$. Should one of these be an equality?

Plainly $B_2(p) = 2$ iff p is a Wieferich prime $2^{p-1} \equiv 1 \bmod p^2$. But what about upper bounds? Can you beat $B_2(p) \ll p^{2-\frac{575}{3276}+\epsilon}$ (Cochrane & P.)?

(c) **An algorithm to compute $\lambda(G)$:** For a given finite group G is there a way of reducing the determination of $\lambda(G)$ to a finite computation? This is true, for example, if all the $n_i = 2, 3$ or 4, but what about a general finite group?

Means of zeros and growth of integer polynomials

(14) (Igor Pritsker) Schur considered polynomials $P_n(z) = a_n \prod_{k=1}^n (z-\alpha_k) \in \mathbb{Z}[z]$ with simple zeros in the closed unit disk D, and studied the arithmetic means of zeros under the additional condition that the leading coefficient a_n is fixed. It was shown in [3] that the zeros α_k are equidistrbuted near the unit circumference under Schur's assumptions. A general discrepancy result for this equidistribution (cf. Theorem 3.1 of [3]) gives the following estimates. If $|a_n| \le M$ and $|\alpha_k| \le 1,\ k = 1, \dots, n$, are simple roots of an integer polynomial P_n, then

$$\left| \frac{1}{n} \sum_{k=1}^n \alpha_k \right| \le 8 \sqrt{\frac{\log n}{n}}, \quad n \ge \max(M, 55),$$

see Corollary 3.2 of [3], and there is $c_1 > 0$ such that

$$\max_D |P_n| \le e^{c_1 \sqrt{n} \log n}, \quad n \ge \max(M, 2),$$

see Corollary 3.3 of [3].

Problem. *Investigate sharpness of the above estimates.*

Let p_m be the mth prime number and consider

$$Q_n(z) := \prod_{m=1}^k \frac{z^{p_m}-1}{z-1} = \prod_{m=1}^k \sum_{j=0}^{p_m-1} z^j, \quad k \in \mathbb{N},$$

where the degree of Q_n is $n = \sum_{m=1}^k (p_m - 1)$. It is shown in [4, p. 157] that Q_n satisfies

$$\left| \frac{1}{n} \sum_{k=1}^n \alpha_k \right| \ge \frac{c_2}{\sqrt{n \log n}},\ c_2 > 0,$$

and

$$\max_D |Q_n| = Q_n(1) = \prod_{m=1}^k p_m \ge e^{c_3 \sqrt{n \log n}},\ c_3 > 0.$$

This shows sharpness of the bounds in question up to logarithmic factors. Several closely related problems were considered in [1].

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Approximation of algebraic numbers

(15) (Damien Roy) Let $\gamma = (1+\sqrt{5})/2 = 1.618\ldots$ A number $\xi \in \mathbb{R}$ is *extremal* if $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$ and there is some $c > 0$ such that the system

$$\begin{cases} |x_0| \le X \\ |x_0\xi - x_1| \le cX^{-1/\gamma} \\ |x_0\xi^2 - x_2| \le cX^{-1/\gamma} \end{cases}$$

has a solution $\underline{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$ for each real number $X \ge 1$.

Known facts:

- The set of these numbers is countably infinite.
- One can't do better than $1/\gamma$ (Davenport and Schmidt, Acta Arith. 1969).
- There is a weak p-adic analogue: Zelo (PhD thesis, arXiv 2008), Bel (JNT, 2013), Bugeaud (PAMS, 2010).

Problem 1. Are such numbers badly approximable ($|\xi - p/q| \ge c/q^2$)?

Problem 2. Is there an exact p-adic analog?

Comments: My impression is that the answer should be 'yes' for the first problem and 'no' for the second. Both problems have a discrete dynamical system flavor. To discuss Problem 1, we need the following notation from (Roy, On the continued fraction expansion of a class of numbers, in: Diophantine approximation, 347361, Dev. Math., 16, SpringerWienNewYork, Vienna, 2008).

A *Fibonacci sequence* in a monoid $(M, *)$ is a sequence $(w_i)_{i\ge1}$ of elements of M such that $w_{i+2} = w_{i+1} * w_i$ for each $i \ge 1$.

In the monoid of words E^* on an alphabet $E = \{a, b\}$ of two letters, with product given by concatenation, we form the Fibonacci sequence $(w_i)_{i\ge1}$ starting with $w_1 = b$ and $w_2 = a$. Its elements $w_3 = ab$, $w_4 = aba$, ... are prefixes of the infinite word $w_\infty = abaababa\ldots$

The set $\mathcal{P}$ of two by two matrices with integers coefficients, non-zero determinant and content one (the gcd of their entries) is a monoid for the product $A * B = \text{content}(AB)^{-1}AB$. A Fibonacci sequence $(W_i)_{i\ge1}$ in $\mathcal{P}$ is said to be *admissible* if there exists $N \in \mathcal{P}$ with $N \ne \pm N^t$ such that W_iN_i is symmetric for each $i \ge 1$, where $N_i = N$ if i is even, and $N_i = N^t$ if i is odd (this is a mild condition because, if it holds for $i = 1, 2, 3$, it holds for all $i \ge 1$).

It can be shown that, for each unbounded admissible Fibonacci sequence $(W_i)_{i\geq 1}$ in $\mathcal{P}$ with $\|W_{i+2}\| \asymp \|W_{i+1}\|\,\|W_i\|$ and $|\det(W_i)| \asymp 1$, there is a unique extremal number ξ such that $\|(\xi,-1)W_i\| \asymp \|W_i\|^{-1}$. Moreover, all extremal numbers arise in this way.

Given $(W_i)_{i\geq 1}$ and ξ as above, consider the morphism of monoids $\mathbb{P}hi\colon E^* \to \mathcal{P}$ mapping w_i to W_i for each $i \geq 1$. Then ξ is badly approximable if there exists a constant C such that $|\det(\mathbb{P}hi(u))| \leq C$ for each prefix u of w_∞. Does there always exist such a constant C?

Galois action on the Berkovich Projective Line

(16) (Robert Rumely) Find the fixed locus of the galois action on the Berkovich Projective Line.

More precisely, let K_v be a non-Archimedean local field, and let $\mathbb{C}_v$ be the completion of the algebraic closure of K_v. Let $\mathbb{P}^1(\mathbb{C}_v)$ be the Berkovich projective line over $\mathbb{C}_v$. The group of continuous automorphisms $\mathrm{Aut}(\mathbb{C}_v/K_v)$ (which is isomorphic to the galois group $\mathrm{Gal}(\overline{K}_v/K_v)$), acts on $\mathbb{P}^1(\mathbb{C}_v)$ in a natural way. Namely, an automorphism σ takes a type II point corresponding to a disc $D(a,r)$ to the type II point corresponding to $D(\sigma(a),r)$. This action on type II points extends by continuity to an action on the whole space, and is well-defined and independent of the choice of coordinates. It is compatible with the usual action of $\mathrm{Aut}(\mathbb{C}_v/K_v)$ on $\mathbb{P}^1(\mathbb{C}_v)$.

The problem is to determine the fixed locus of $\mathrm{Aut}(\mathbb{C}_v/K_v)$, acting on $\mathbb{P}^1(\mathbb{C}_v)$. An answer would be useful in many questions in arithmetic dynamics.

Discussion. The fixed locus is clearly path-connected (hence is a tree); it is closed, and contains the tree spanned by $\mathbb{P}^1(K_v)$. However, it can be strictly larger than the tree spanned by $\mathbb{P}^1(K_v)$: for example, when $K_v = \mathbb{Q}_2$, the point corresponding to the disc $D(\sqrt{-1}, 1/2)$ is fixed by the automorphism group, but does not belong to the tree spanned by $\mathbb{P}^1(\mathbb{Q}_2)$.

Let k_v be the residue field of the ring of integers of K_v, and let q_v be its order. The group $\mathrm{Aut}(\mathbb{C}_v/K_v)$ acts on the tangent space at any type II point in fixed locus, and it fixes exactly $q_v + 1$ tangent directions. This means the fixed locus is not open in $\mathbb{P}^1(\mathbb{C}_v)$ (for either the weak or strong topology).

Beyond this, not much is known.

Here are some subsidiary questions that could be asked:

1) When p is odd, and $K_v = \mathbb{Q}_p$, find examples of points in the fixed locus which lie outside the tree spanned by $\mathbb{P}^1(\mathbb{Q}_p)$. Are there any? It is possible that when p is odd, the fixed locus is just the tree spanned by $\mathbb{P}^1(\mathbb{Q}_p)$? When $p = 2$, are there any examples beside the one above? The construction of points outside the tree spanned by $\mathbb{P}^1(K_v)$ basically involves finding algebraic numbers whose conjugates lie closer to each other than to points of K_v.

2) Is the fixed locus tree discretely branched (like the tree spanned $\mathbb{P}^1(K_v)$), or is it densely branched (like $\mathbb{P}^1(\mathbb{C}_v)$ itself)? Does the fixed locus contain any type IV points?

3) Does the fixed locus lie within a bounded distance (for the logarithmic path-distance metric) of the tree spanned by $\mathbb{P}^1(K_v)$?

An infinite residue sequence

(17) (Andrzej Schinzel) Given a number field K, does there exist an infinite sequence of integers of K: $a_1, a_2, \ldots$, such that for every ideal $\mathfrak{a}$ of K among first $N\mathfrak{a}$ terms of the sequence there are represented all residues mod $\mathfrak{a}$?

See Problem 8 in the paper of W. Narkiewicz, *Open problems*, Bull. SMF, Mémoire 25 (1971), p. 161, and A. Schinzel, *Selecta*, vol. 2, p. 1370, Problem 30. See also

Adam, David; Cahen, Paul-Jean. Newtonian and Schinzel quadratic fields. J. Pure Appl. Algebra 215 (2011), no. 8, 1902–1918.

Literature: MR 47#3343, 50#2115, 51#5549, 54#5174.

Nonzero coefficients of cyclotomic polynomials

(18) (Andrzej Schinzel) Given a squarefree number n, estimate from below the number of non-zero coefficients of the cyclotomic polynomial of order n.

See J. H. Conway and A. J. Jones, Acta Arith. 30 (1976), 229–240, and A. Schinzel, *Selecta*, vol. 2, p. 1370, Problem 31.

An $\mathbf{ax+by+cz=0}$ conjecture

(19) (Chris Smyth) The following conjecture appeared in [1].

Conjecture. We are given positive integers a, b, c, pairwise coprime and satisying $a \le b+c$, $b \le c+a$ and $c \le a+b$. Then there exists a finite number, n say, of integer solutions $(x_i, y_i, z_i) \neq (0,0,0)$ $(i = 1, \ldots, n)$ of $ax + by + cz = 0$ such that

$$\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\} = \{z_1, \ldots, z_n\}, \qquad (*)$$

as multisets.

Lemma. Assuming (as wlg we can) that $\gcd(a, b, c) = 1$ then each of the conditions on a, b, c in the conjecture is necessary.

Proof. Suppose that $a > b + c$ and, wlg, that x_1 has maximal modulus among all the x_i's. Then, by (*), also $|y_1| \le |x_1|$ and $|z_1| \le |x_1|$. Note that $x_1 \neq 0$. Hence

$$a|x_1| = |-ax_1| = |by_1 + cz_1| \le (b + c)|x_1|,$$

so that $a \le b + c$. Similarly $b \le c + a$ and $c \le a + b$.

Now take a prime p, and among the x_i assume wlg that x_1 is divisible by the smallest power of p (i.e., it has maximal p-adic valuation). Then, by (*), $|y_1|_p \le$

$|x_1|_p$ and $|z_1|_p \le |x_1|_p$. Hence $|ax_1| = |-by_1 - cz_1|_p \le \max(|b|_p, |c|_p)|x_1|_p$ and so, since $x_1 \ne 0$, $|a|_p \le \max(|b|_p, |c|_p)$. Since $\gcd(a,b,c) = 1$, b and c cannot both be divisible by p. Doing this for all primes p, we have $\gcd(b,c) = 1$. Similarly $\gcd(c,a) = \gcd(a,b) = 1$.

Applications.

Theorem$^+$. Suppose the conjecture holds. Then the equation $ax + by + cz = 0$ has a solution in nonzero conjugate algebraic integers x, y, z iff a, b and c satisfy the conditions in the conjecture.

Proof. Suppose that α_1, α_2 and α_3 are, conjugate, nonzero, and that $a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$. Let N be a normal extension of $\mathbb{Q}$ containing α_1 By applying to $a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$ an automorphism of N that takes α_1 to a conjugate having maximal modulus, we can assume, after relabelling the conjugates of α_1, that α_1 itself has maximal modulus. Then $a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$ gives

$$a|\alpha_1| = |b\alpha_2 + c\alpha_3| \le (a+b)|\alpha_1|,$$

so that $a \le b + c$. Similarly $b \le c + a$ and $\le a + b$.

A p-adic argument, replacing $|.|$ by $|.|_p$ in the above, gives $|a|_p \le \max(|b|_p, |c|_p)$. Hence, assuming wlg that $\gcd(a,b,c) = 1$, b and c cannot both be divisible by p. doing this for all primes p, we obtain $\gcd(b,c) = 1$. Similiarly, $\gcd(c,a) = \gcd(a,b) = 1$. Hence a, b and c must satisfy all the conditions of the conjecture.

Conversely, assume that a, b and c satisfy all the conditions of the conjecture, and that the conjecture holds. Take integer solutions (x_i, y_i, z_i) of $ax + by + cz = 0$, as in the conjecture. Then for any numbers β_i, we have $a(\sum_i x_i\beta_i) + b(\sum_i y_i\beta_i) + c(\sum_i z_i\beta_i) = 0$. Now take the β_i to be the roots of $z^n - z - 1 = 0$. The splitting field of this equation has Galois group the full symmetric group S_n. Then condition (*) implies that $\sum_i x_i\beta_i$, $\sum_i y_i\beta_i$ and $\sum_i z_i\beta_i$ are conjugate algebraic integers.

A similar (multiplicative) result concerns solving the equation $x^a y^b z^c = 1$ in conjugate algebraic integers, not roots of unity. (The equation always has a solution in conjugate roots of unity.)

Theorem$^\times$. Suppose the conjecture holds. Then the equation $x^a y^b z^c = 1$ has a solution in conjugate algebraic integers x, y, z that are not roots of unity iff a, b and c satisfy the conditions in the conjecture.

The proof is very similar to the additive case. In the converse part of the proof, we have $(\prod_i \beta_i^{x_i})^a \cdot (\prod_i \beta_i^{y_i})^b \cdot (\prod_i \beta_i^{z_i})^c = 1$, where $\prod_i \beta_i^{x_i}$, $\prod_i \beta_i^{y_i}$ and $\prod_i \beta_i^{z_i}$ are conjugate.

Generalisation to k variables.

General Conjecture. We are given positive integers $a_1, \dots, a_k$, such that the gcd of any $k-1$ of them is 1 and and such that the sum of any $k-1$ of them is at least as large as the remaining one. Then there exists a finite number, n say, of integer solutions $(x_{i1}, x_{i2}, \dots, x_{ik}) \neq (0, 0, \dots, 0)$ $(i = 1, \dots, n)$ of $a_1x_1 + \cdots + a_kx_k = 0$ such that

$$\{x_{1j}, \dots, x_{nj}\}$$

is the same multiset for all $j = 1, 2, \dots, k$.

References

[1] Smyth, C. J. Additive and multiplicative relations connecting conjugate algebraic numbers. J. Number Theory **23** (1986), 243–254.

Sets of Pisot-like numbers

(20) (Chris Smyth) Let $r > 0$, and S_r be the set of real positive algebraic integers $\alpha > r$, all of whose conjugates $\neq \alpha$ are $< r$. Clearly S_1 is the set of Pisot numbers – known to form a closed subset of $\mathbb{R}$ (Salem 1944).

Lemma 1. If $r < 1$ then S_r is discrete (and so certainly closed). Further, $\cup_{r<1} S_r = S_1$.

Proof. Let n denote the degree of $\alpha \in S_r$, where $r < 1$. Then

$$1 \leq |\operatorname{Norm}(\alpha)| \leq \alpha r^{n-1},$$

so that $\alpha > \left(\frac{1}{r}\right)^{n-1}$. Hence the degree of such α in $(1, B]$ is at most $1 + \log B / \log\left(\frac{1}{r}\right)$. Since there are only a finite number of monic integer polynomials of degree at most n having all zeros in the disc $|z| \leq B$, we have that there are only finitely many α in $S_r \cap (1, B]$. Thus S_r is discrete.

The final statement is obvious.

Lemma 2. The union $\cup_{r>0} S_r$ is dense in $(1, \infty)$.

Proof. For positive integers n, b, let $p(z) = z^{2n+1}(z^{2n} - b) - 1$. Then p has a unique zero $\alpha > 1$, and all other zeros have smaller modulus than α. Thus α is a Perron number, and so it belongs to S_α.

Now take any $t > 1$, and put $b = \lfloor t^{2n} \rfloor$. Then as $n \to \infty$ for fixed t, the largest root of $p(z) = z^{2n+1}(z^{2n} - \lfloor t^{2n} \rfloor) - 1$ tends to t. Hence $\cup_{r>0} S_r$ is dense in $(1, \infty)$.

Question. For $r > 1$, is S_r dense in $(1, \infty)$? Is it closed? Or what?

Beyond Lehmer's problem for α with $\mathbb{Q}(\alpha)$ Galois

(21) (Chris Smyth; discussion by Francesco Amoroso follows.)

Amoroso and David [1] proved that the answer to Lehmer's question is 'yes' in the case of algebraic numbers α of degree d with $\mathbb{Q}(\alpha)$ a Galois extension of $\mathbb{Q}$. That is, there is then a constant $C > 1$ such that the Mahler measure $M(\alpha)$ satisfies $M(\alpha) \geq C$ when $M(\alpha) > 1$. But does something stronger hold in this case? Might there even be a constant $c > 1$ such that $M(\alpha) \geq c^d$ when $M(\alpha) > 1$?

Now such a $\mathbb{Q}(\alpha)$ is either totally real or totally nonreal. In the totally real case, we have, by a result of Schinzel [2], that the stronger result does hold, with constant $c = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2} = 1.272\ldots$. Here the fact that $\mathbb{Q}(\alpha)$ is Galois is not used. So we can confine our attention to the totally nonreal case – i.e., the problem is already half solved!

Let $n \geq 2$ and $\beta_1, \ldots, \beta_n$ be the roots of $z^n - z - 1$, known to be irreducible for all n, and to have Galois group the full symmetric group S_n. Put

$$\alpha = \beta_1^1 \beta_2^2 \ldots \beta_{n-1}^{n-1}.$$

Then the Galois closure of $\mathbb{Q}(\beta_1)$ is $\mathbb{Q}(\alpha)$ of degree $d = n!$ over $\mathbb{Q}$. The following table shows the value of $M(\alpha)^{1/d}$ for $n = 2, \ldots, 9$, computed with Maple.

n	$d = n!$	$M(\alpha)^{1/d}$
2	2	1.2720196495
3	6	1.1509639252
4	24	1.2428334720
5	120	1.2292495215
6	720	1.2846087150
7	5040	1.2833028970
8	40320	1.3243452986
9	362880	1.3307248410

(a) Does anyone know of any smaller values of $M(\alpha)^{1/d} > 1$ for α of degree d with $\mathbb{Q}(\alpha)$ Galois?

(b) Does the above sequence of values $M(\alpha)^{1/d}$ tend to a limit as $n \to \infty$ and, if so, what is it?

References

[1] Amoroso, Francesco; David, Sinnou. Le problème de Lehmer en dimension supérieure. J. Reine Angew. Math. 513 (1999), 145–179.

[2] Schinzel, A. On the product of the conjugates outside the unit circle of an algebraic number. Acta Arith. 24 (1973), 385–399. Addendum: Acta Arith. 26 (1974/75), no. 3, 329–331.

On Galois Lehmer's problem

(22) (Francesco Amoroso) Some participants are interested in issues related to Lehmer's problem in Galois extensions. More specifically, Chris Smyth asks (see Problem 21

above) for lower bounds for the height of a non-zero algebraic number α, not a root of unity, such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension.

As a corollary of our main theorem, we proved in [1] that in this situation Lehmer's conjecture holds: the height of α is bounded from below by c/D where c is a positive, effective, constant and $D = [\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Very recently, D. Masser observed that this lower bound can be strengthened, combining the quoted article and the relative Dobrowolski's Theorem of [2].

Fact 1. *Let $\mathbb{F}/\mathbb{Q}$ be a Galois extension of degree D. Then, for any $\alpha \in \mathbb{F}^*$ which is not a root of unity and for any $\varepsilon > 0$ there exists a positive $c(\varepsilon)$ such that:*

$$h(\alpha) \geq c(\varepsilon) D^{-1/2-\varepsilon} .$$

This result will appear as an exercise (with the additional, but inessential to the proof, assumption $\mathbb{F} = \mathbb{Q}(\alpha)$) in the forthcoming book [3]. Here $h(\alpha)$ is the *Weil height* of α, related to its Mahler measure $M(\alpha)$ by $\log M(\alpha) = \deg(\alpha) h(\alpha)$.

We remark that the above lower bound is optimal: take for $\mathbb{F}$ the splitting field of $x^d - 2$ and $\alpha = 2^{1/d}$. Nevertheless, this result can be further strengthened for a *generator* α of a Galois extension. Indeed we can prove:

Fact 2. *Let $\alpha \in \overline{\mathbb{Q}}^*$ of degree D, not a root of unity, such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. Then for any $\varepsilon > 0$ there exists a positive $c(\varepsilon)$ such that:*

$$h(\alpha) \geq c(\varepsilon) D^{-\varepsilon} .$$

References

[1] F. Amoroso and S. David, "Le problème de Lehmer en dimension supérieure", *J. Reine Angew. Math.* **513** (1999), 145–179.

[2] F. Amoroso and U. Zannier, "A relative Dobrowolski's lower bound over abelian extensions." *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), no. 3, 711–727.

[3] D. Masser, "Auxiliary Polynomials in Number Theory". In Press.

Polynomials with low height and prescribed vanishing

(23) (Jeff Vaaler) The following result was proved E. Bombieri and J. D. Vaaler in "Polynomials with low height and prescribed vanishing", *Analytic Number Theory and Diophantine Problems*, Progress in Mathematics, Volume 70, edited by A.C. Adolphson, J.B. Conrey, A. Ghosh, R.I. Yager, Birkhäuser, 1987, pp. 53–73:

Theorem 3. *Let $\delta > 0$ and let $P(x)$ be a nonzero polynomial in $\mathbb{Z}[x]$ with*

$$\deg P = M \text{ and } 1 \leq \mu(P) \leq 2 - \delta,$$

where $\mu(P)$ is the Mahler measure of P. Then there exists a monic polynomial

$$Q(x) = x^N + \epsilon_1 x^{N-1} + \epsilon_2 x^{N-2} + \cdots + \epsilon_N$$

such that:

(i) $\epsilon_n \in \{-1, 0, 1\}$ *for each* $n = 1, 2, \ldots, N$,

(ii) $N \ll_\delta M^2 \log M$,
(iii) $P(x)$ *divides* $Q(x)$ *in the ring* $\mathbb{Z}[x]$.

Using this theorem it is possible to prove the existence of a polynomial $Q(x)$ in $\mathbb{Z}[x]$ having coefficients restricted to the set $\{-1, 0, 1\}$, and a zero of multiplicity 4 that is not a root of unity. To accomplish this we apply the theorem to

$$P(x) = \left(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1\right)^4,$$

for which we find that

$$\mu(P) = (1.17628082\cdots)^4 = 1.91445015\dots.$$

But we also have

$$(1.17628082\cdots)^5 = 2.251931003\dots,$$

so this method *fails* to establish the existence of a polynomial $Q(x)$ in $\mathbb{Z}[x]$ with a zero of multiplicity 5 that is not a root of unity.

These observations suggest the following question: for each positive integer L, does there exist a monic polynomial $Q(x)$ in $\mathbb{Z}[x]$ such that the coefficients of Q are in the set $\{-1, 0, 1\}$, and Q has a zero of multiplicity at least L at a point that is not zero, and is not a root of unity? We do not know the answer to this question if $L \geq 5$.

Bounds relating field discriminant and minimal height of a field generator

(24) (Martin Widmer) The questions below stem from Ruppert [1] and Vaaler and Widmer [3]. Let L be a number field of degree $D > 1$ and absolute discriminant Δ_L. For $\alpha \in L$ let

$$H(\alpha) = \prod_{v \in M_L} \max\{1, |\alpha|_v\}^{\frac{d_v}{D}}$$

be the absolute multiplicative Weil height of α, and let

$$\delta(L) = \inf\{H(\alpha); L = \mathbb{Q}(\alpha)\}$$

be the smallest height of a generator. We consider the set of values

$$\frac{\log \delta(L)}{\log |\Delta_L|}$$

as L runs over all number fields of fixed degree $D > 1$. What are the cluster points of this set? Ruppert [1] has shown that

$$\lim \inf_{[L:\mathbb{Q}]=D} \frac{\log \delta(L)}{\log |\Delta_L|} = \frac{1}{2D(D-1)}.$$

What about the largest cluster point (cf. [2, Question 2])? Is it true that

$$\lim \sup_{[L:\mathbb{Q}]=D} \frac{\log \delta(L)}{\log |\Delta_L|} \leq \frac{1}{2D}?$$

This is known to be true if either $D = 2$ [1,Proposition 2] or if D is odd [2,Theorem 1.2]. It is also known to be true under GRH [2, Theorem 1.3]. Is it true that for $D > 3$

$$\lim \sup_{[L:\mathbb{Q}]=D} \frac{\log \delta(L)}{\log |\Delta_L|} > \frac{1}{2D(D-1)}?$$

We know that the answer is "yes" if D is composite [3, Theorem 1.2] or if $D = 5$ or under a very weak form of a folk conjecture about the distribution of number fields [3, Theorem 1.3]. What if $D = 3$?

However, so far only one cluster point of the aforementioned set has been located.

References

[1] W. Ruppert, Small generators of number fields, Manuscripta Math., **96** (1998), 17-22.

[2] J. D. Vaaler and M. Widmer, A note on small generators of number fields, Diophantine Methods, Lattices, and Arithmetic Theory of Quadratic Forms, Contemporary Mathematics, vol. 587, Amer. Math. Soc., Providence, RI, 2013.

[3] ______ ,______ , Number fields without small generators, to appear in Math. Proc. Cam. Philos. Soc., doi:10.1017/S0305004115000298.