Optimization of Lyapunov Exponents

RDS and MET Workshop at BIRS, Banff, Canada

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Joint work with

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Plan of the talk

1. Thanks to the organizers.
2. “Classical” ergodic optimization
3. Non-commutative ergodic optimization: what we would like to do
4. Setting: One-step cocycles and domination
5. A useful tool: Barabanov functions
6. The main result
7. Strategy of the proof

ERGODIC OPTIMIZATION:

FROM BIRKHOFF AVERAGES TO LYAPUNOV EXPONENTS

Ref.: Survey by O. Jenkinson (DCDS, 2006)

“Classical” ergodic optimization

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Given a continuous function $f\colon X \to \mathbb{R}$, let

$$M(f) := \sup_{\mu \in \mathcal{M}(T)} \int f \, d\mu.$$

An invariant measure for which the sup is attained is called *maximizing* for f. (They always exist.)

“Classical” Ergodic Optimization is concerned about determining $M(f)$, describing the maximizing measures μ, their properties etc.

Rem.: Connections with Classical Mechanics (Aubry–Mather. . .), Thermodynamical Formalism (zero temperature limits), Multifractal Analysis . . .

The place where maximizing measures live

Theorem (Many authors(*), 90's, 00's)

Suppose $T: X \to X$ is "hyperbolic" and f is Hölder continuous. Then there exists a compact T-invariant set $K \subset X$ (called a "Mather set") such that:

$$\mu \in \mathcal{M}(T) \text{ is maximizing} \quad \Leftrightarrow \quad \operatorname{supp} \mu \subset K\,.$$

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N.B.: The result is false for C^0 functions; in this case there are examples where the maximizing measure is unique and has full support.

Proof of the existence of Mather sets

Suppose $\tilde{f}$ is cohomologous to f, i.e.,

$$\tilde{f} = f + h \circ T - h \quad \text{for some } h$$

Then

$$M(f) = \sup_{\mu \in \mathcal{M}(T)} \int f \, d\mu = \sup_{\mu \in \mathcal{M}(T)} \int \tilde{f} \, d\mu = M(\tilde{f}).$$

$$\mu \text{ is maximizing for } f \quad \Longleftrightarrow \quad \mu \text{ is maximizing for } \tilde{f}.$$

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Theorem (Mañé(*) Lemma or Revelation Lemma)

Suppose $T \colon X \to X$ is "hyperbolic" and f is Hölder continuous. Then $\exists\ \tilde{f}$ cohomologous to f such that $\tilde{f} \le M(\tilde{f})$.

Then the maximizing measures are exactly those who live in the (Mather set) $K := \{x \in X;\ \tilde{f}(x) = M(\tilde{f})\}$.

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If T is "expanding" then for ***generic*** *Lipschitz functions f:*

- ***(several authors)*** *There is a* ***unique*** *maximizing measure.*

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Related results on periodicity of maximizing measures by:

Hunt–Ott'96 (experimental), Yuan–Hunt'99, Contreras–Lopes-Thieullen'01, Bousch'01, Quas–Siefken'12. The latter deals with spaces of *super-continuous* functions over the shift.

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Theorem (B.–Zhang, preprint ArXiv 2015)

*On spaces of sufficiently super-continuous functions, the conclusion of Quas–Siefken–Contreras is not only topologically generic but **prevalent** (has "full probability").*

Non-commutative Ergodic Optimization

What if instead of optimizing integrals (Birkhoff averages), we optimize Lyapunov exponents instead?

Lyapunov exponents (2×2 case)

Let $T\colon X \to X$ be a homeomorphism of a compact space. Let $A\colon X \to \mathrm{GL}(2,\mathbb{R})$ be continuous. The pair (T, A) is called a *cocycle*. Consider "Birkhoff-like" products:

$$A^{(n)}(x) := A(T^{n-1}x)\cdots A(Tx)A(x).$$

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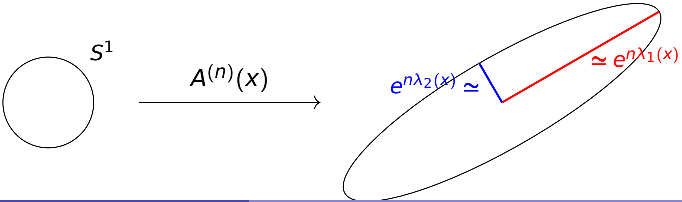
The *upper and lower Lyapunov exponents* at the point x (if they exist) are:

$$\lambda_1(x) := \lim_{n\to+\infty} \frac{1}{n} \log \left\| A^{(n)}(x) \right\|$$

$$\lambda_2(x) := \lim_{n\to+\infty} \frac{1}{n} \log \left\| [A^{(n)}(x)]^{-1} \right\|^{-1}$$



Oseledets Theorem

Theorem (Invertible 2 × 2 Oseledets)

Let μ be an ergodic probability measure for T. Then there exist $\lambda_1(\mu) \geq \lambda_2(\mu)$ such that $\lambda_i(x) = \lambda_i(\mu)$ for μ-a.e. x.

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Moreover, if $\lambda_1(\mu) > \lambda_2(\mu)$ then there exists an "Oseledets splitting" $\mathbb{R}^2 = E_1(x) \oplus E_2(x)$, defined for μ-a.e. x and such that for each $i = 1, 2$:

- *the spaces E_i vary measurably w.r.t. x;*
- *the spaces E_i are equivariant: $A(x)(E_i(x)) = E_i(Tx)$;*
- $\displaystyle\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^{(n)}(x) v\| = \lambda_i(\mu), \quad \forall v \in E_i(x) \setminus \{0\}$ *($i = 1, 2$).*

Optimization of Lyapunov exponents

Fixed cocycle (T, A) as above, let:

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- How are those measures? What are their entropies, dimensions, etc?
- Are they unique?
- Are they characterized by their support?

Summary of the main result (B., Rams)

We will exhibit a natural open class of 2×2 cocycles for which the Lyapunov-maximizing and -minimizing measures have zero entropy.

An important source of ideas:

paper by T. Bousch, J. Mairesse (JAMS, 2002).

A SPECIAL SITUATION:

ONE-STEP COCYCLES

One-step cocycles

A cocycle (T, A) is *one-step* if:

- $T : X \to X$ is the two-sided full shift on $k \geq 2$ symbols, so

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- $A \colon X \to \mathrm{GL}(2, \mathbb{R})$ only depends on the zeroth symbol. In other words, there is a list of matrices $A_1, \dots, A_k$ such that

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Equivalent setting: *Linear IFS's (iterated function systems)*.

Joint spectral radius and subradius

Equivalent *elementary* definitions of $\lambda_1^{\top}$, $\lambda_1^{\perp}$ **for one-step cocycles**:

$$\lambda_1^{\top} = \lim_{n\to\infty} \frac{1}{n} \log \sup_{i_1,\dots,i_n} \|A_{i_1} \dots A_{i_n}\| \,,$$
$$\lambda_1^{\perp} = \lim_{n\to\infty} \frac{1}{n} \log \inf_{i_1,\dots,i_n} \|A_{i_1} \dots A_{i_n}\| \,.$$

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For one-step cocycles, the numbers $e^{\lambda_1^\top}$ and $e^{\lambda_1^\perp}$ are known as *joint spectral radius* (Rota, Strang 1960) and *joint spectral subradius* (Gurvits, 1995) of the set of matrices $\{A_1, \dots, A_k\}$.

Applications in wavelets, control theory ...

The finiteness conjecture on the Joint spectral radius

Finiteness conjecture (1995): *For one-step cocycles, there always exists a Lyapunov-maximizing measure supported on a periodic orbit.*

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The conjecture was shown to be **false** by Bousch and Mairesse (2002); their counterexamples are pairs of matrices in $GL(2, \mathbb{R})$ such that the maximizing measures are **sturmian** but not periodic.

Domination for one-step cocycles

An one-step cocycle is called *dominated* if the Lyapunov exponents are uniformly separated, for all invariant measures. In other words,

$$(\lambda_1 - \lambda_2)^{\perp}(A) := \inf_{\mu \in \mathcal{M}_{\mathrm{erg}}(T)} (\lambda_1(A, \mu) - \lambda_2(A, \mu))$$

is positive.

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Example (Perron–Frobenius theorem)

Each A_i is an (entrywise) **positive** matrix ⇒ the cocycle is dominated.

Domination for one-step cocycles: notes

Recall: An one-step cocycle is called *dominated* if $(\lambda_1 - \lambda_2)^\perp > 0$.

- For one-step cocycles (only!), the definition above is equivalent to the more usual condition that the one Oseledets direction “dominates” the other.

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- In the area-preserving case (2×2 matrices with determinant ± 1), domination is equivalent to the familiar notion of *uniform hyperbolicity*.
- Domination is a natural weakening of uniform hyperbolicity and is important smooth dynamics (e.g. in partial hyperbolicity).
- It’s also weaker than exponential dichotomy.

Domination for one-step cocycles: geometrical viewpoint

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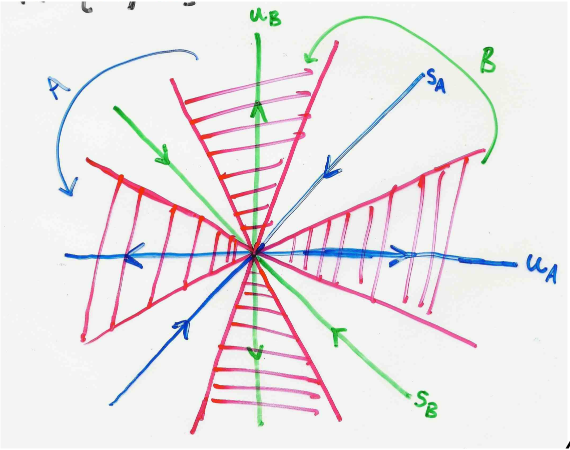
Theorem (Avila, B., Yoccoz)

An one-step cocycle is dominated IFF it has a multicone.

Consequence: domination is an open condition. (This is also true for more general cocycles.)

The simplest example where the multicone cannot be taken as a cone is:



$A_1 = A$, $A_2 = B$.

Examples with complicated combinatorics: see [Avila–B.–Yoccoz]

BARABANOV FUNCTIONS

FOR DOMINATED ONE-STEP COCYCLES:

An analogue of the Mañé revelation lemma

Barabanov functions

Assume given an one-step dominated cocycle, with a multicone M. Let $\overrightarrow{M} := \{v \in \mathbb{R}^2;\ v = 0 \text{ or } \mathbb{R} \cdot v \in M\}$ be the "support" of M.

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Proposition (Existence of a Barabanov function)

There exists a continuous function $|\!|\!|\cdot|\!|\!| : \overrightarrow{M} \to [0, \infty)$ *such that for every* $v \in \overrightarrow{M}$, $t \in \mathbb{R}$ *we have*

- $|\!|\!|tv|\!|\!| = |t|\ |\!|\!|v|\!|\!|$ *(homogeneity);*

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- $|||tv||| = |t|\ |||v|||$ *(homogeneity);*
- $\max_{i \in \{1,\dots,k\}} |||A_i v||| = e^{\lambda_1^\top} |||v|||$ *(main property).*

We call $|||\cdot|||$ an *upper Barabanov function*.

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Consequence: For any $v \in \overrightarrow{M}$, we can find recursively symbols $i_1, i_2, \ldots \in \{1, \ldots, k\}$ such that

$$|||A_{i_n} \cdots A_{i_1}(v)||| = e^{n\lambda_1^\top} |||v|||.$$

Barabanov functions

Upper Barabanov function:

$$|||tv||| = |t|\ |||v|||, \qquad \max_{i\in\{1,\dots,k\}} |||A_i v||| = e^{\lambda_1^{\top}} |||v|||.$$

- Actually (even without domination) there is a true norm $|||\cdot|||$ in $\mathbb{R}^2$ with the properties above; it is called *Barabanov norm* (1988).

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- Actually (even without domination) there is a true norm $|||\cdot|||$ in $\mathbb{R}^2$ with the properties above; it is called *Barabanov norm* (1988).
- There is also a *lower Barabanov function* $|||\cdot|||'$ such that

$$|||tv|||' = |t|\ |||v|||', \qquad \min_{i \in \{1,\dots,k\}} |||A_i v|||' = e^{\lambda_1^\perp} |||v|||' .$$

  In this case the domination hypothesis is really needed.
- Higher dimension: see [B.–Morris, Proc. LMS (to appear)].

Proof of existence of Barabanov functions

We apply Schauder fixed point theorem to an appropriate space of Lispchitz functions (with respect to the Hilbert metric) on the multicone . . .

BTW, this proof also gives a Lipschitz property for $|||\cdot|||$, which is also useful.

Mather sets

Proposition

If an one-step cocycle is ***dominated*** *then there are nonempty compact shift-invariant sets* $K^\top, K^\perp \subset k^{\mathbb{Z}}$ *s.t. for every shift-invariant measure* μ*,*

$$\operatorname{supp}\mu \subset K^\top \iff \lambda_1(\mu) = \lambda_1^\top, \qquad \operatorname{supp}\mu \subset K^\perp \iff \lambda_1(\mu) = \lambda_1^\perp$$

(In particular, there exist Lyapunov-optimizing measures.)

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Proof:

$$K^\top := \left\{ \xi \in X;\ u \in E_1(\xi),\ n \in \mathbb{Z} \ \Rightarrow\ |||A^{(n)}(\xi)(u)||| = e^{n\lambda_1^\top} |||u||| \right\}.$$

Rem.: The proposition above should also follow from the "classical" (commutative) theory applied to the function $f := \log \|A|E_1\|$. Anyway, our Barabanov functions with be fundamental for what we will do next.

OUR MAIN RESULT

Non-overlapping condition

We say that the matrices $A_1, \dots, A_k$ satisfy *forward nonoverlapping condition* (NOC^+) if they admit a multicone M such that

$$i \neq j \ \Rightarrow\ A_i(M) \cap A_j(M) = \varnothing.$$

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We say that the matrices $A_1, \dots, A_k$ satisfy *backwards nonoverlapping condition* (NOC$^-$) if $\{A_1^{-1}, \dots, A_k^{-1}\}$ satisfy the forward nonoverlapping condition.

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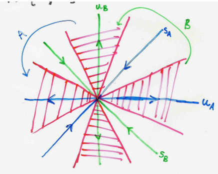
$$i \neq j \;\Rightarrow\; A_i(M) \cap A_j(M) = \varnothing.$$

We say that the matrices $A_1, \ldots, A_k$ satisfy *backwards nonoverlapping condition* (NOC^-) if $\{A_1^{-1}, \ldots, A_k^{-1}\}$ satisfy the forward nonoverlapping condition.

If both conditions hold then we say that the matrices satisfy the *nonoverlapping condition* (NOC).

Example: The one in a previous figure:



The main theorem

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If an ***one-step*** *cocycle is* ***dominated*** *and satisfies the* ***non-overlapping*** *condition*

The main theorem

Theorem (B., Rams)

*If an **one-step** cocycle is **dominated** and satisfies the **non-overlapping** condition then the sets $K^\top$, $K^\perp$ have **zero topological entropy**.*

In particular, the optimizing measures have zero entropy.

Remarks

- The zero entropy measures that appear in the theorem are **not necessarily supported on periodic orbits** (∃ examples by Bousch–Mairesse and others).
- Neither they are necessarily unique.
- The non-overlapping condition is indeed required: for example, if $A_1 = A_2$ then there are optimal measures with positive entropy.
- We think that a more general theorem holds, with $T =$ subshift of finite type, $A =$ locally constant cocycle, but we haven't checked the details.

Questions

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- What about **higher dimension**?

There is much more to be done in this subject!

PROOF OF THE THEOREM

Oseledets directions

Given a bi-infinite word $\xi \in X = k^{\mathbb{Z}}$, split it as $\xi = (\xi_-, \xi_+) \in k^{\mathbb{Z}_-} \times k^{\mathbb{Z}_+}$, where $\mathbb{Z}_- = \{\dots, -2, -1\}$ and $\mathbb{Z}_+ = \{0, 1, \dots\}$.

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The first Oseledets direction only depends on the past, while the second one only depends on the future:

$$E_1(\xi) = E_1(\xi_-), \quad E_2(\xi) = E_2(\xi_+).$$

Some facts about the sets of directions

$$C_1 := \{E_1(\xi_-);\ \xi \in k^{\mathbb{Z}_-}\}, \quad C_2 := \{E_2(\xi_+);\ \xi \in k^{\mathbb{Z}_+}\}.$$

M is a multicone $\quad \Rightarrow \quad C_1 = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\dots,i_n} A_{i_1} \cdots A_{i_n}(M)$ (nested)

Proof: $E_1(\xi_-)$ appears when we take $i_j = \xi_{-j}$.

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forward non-overlapping condition (NOC$^+$)

$\Updownarrow$

$E_1 \colon k^{\mathbb{Z}_-} \to C_1$ **is a bijection**.

$\Downarrow$

C_1 **is a Cantor set**.

Analogously for $E_2 \colon k^{\mathbb{Z}_+} \to C_1$.

Main proposition

Proposition (E_1 determines E_2 a.s.)

Suppose that the cocycle satisfies the forward non-overlapping condition (NOC$^+$).
Let μ be an ergodic measure such that $\lambda_1(\mu) = \lambda_1^\top$ or $\lambda_1^\perp$.

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Let μ be an ergodic measure such that $\lambda_1(\mu) = \lambda_1^\top$ or $\lambda_1^\perp$.
Then for μ-a.e. $\xi \in X$, the direction $E_1(\xi_-)$ is uniquely determined the direction $E_2(\xi_+)$.

Main Proposition ⇒ Theorem

Suppose that the cocycle satisfies the non-overlapping condition ($\mathrm{NOC}^{+} + \mathrm{NOC}^{-}$). Let μ be an ergodic measure supported in $K^{\top}$, that is, $\lambda_1(\mu) = \lambda_1^{\top}$.

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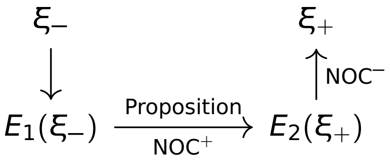

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$$\begin{array}{ccc} \xi_- & & \xi_+ \\ \downarrow & & \uparrow \scriptstyle{\mathrm{NOC}^-} \\ E_1(\xi_-) & \xrightarrow[\mathrm{NOC}^+]{\text{Proposition}} & E_2(\xi_+) \end{array}$$

This implies that $h(\mu) = 0$, for all μ supported in $K^\top$.

By the Entropy Variational Principle, $h_{\mathrm{top}}(K^\top) = 0$. Analogously for $K^\perp$.

This proves the Theorem modulo the Proposition.

PROOF OF THE MAIN PROPOSITION

Cross ratio

Given nonzero vectors $u, v, u', v' \in \mathbb{R}^2$, (no three of them being collinear), we define their *cross ratio*:

$$[u, v; u', v'] := \frac{u \times u'}{u \times v'} \cdot \frac{v \times v'}{v \times u'} \in \mathbb{R} \cup \{\infty\},$$

where $\times$ is the cross product on $\mathbb{R}^2$ (that is, determinant).

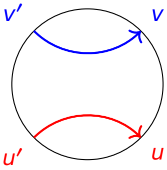
Actually the value above only depends on the directions determined by the vectors. So we can define the cross ratio of four points in $\mathbb{P}^1$.

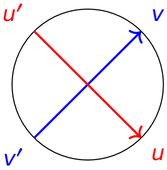
The cross ratio is invariant by linear isomorphisms.

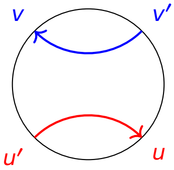
Configurations of four points in $\mathbb{P}^1$

Let $u, v, u', v' \in \mathbb{P}^1$ be distinct. The configuration is called *co-parallel*, *crossing* or *anti-parallel* according to the value of the cross ratio $[u, v; u', v']$ as follows:

Co-parallel	**Crossing**	**Anti-parallel**
$0 < [u, v; u', v'] < 1$	$[u, v; u', v'] > 1$	$[u, v; u', v'] < 0$







Restrictions among the Oseledets directions

Lemma (Geometrical Lemma)

$K^\top$, $K^\perp$ = *Mather sets of an* 1*-step dominated cocycle.*

$$\text{Then} \quad \xi, \eta \in K^\top \quad \Rightarrow \quad \left|[E_1(\xi), E_1(\eta); E_2(\xi), E_2(\eta)]\right| \geq 1\,,$$

$$\xi, \eta \in K^\perp \quad \Rightarrow \quad \left|[E_1(\xi), E_1(\eta); E_2(\xi), E_2(\eta)]\right| \leq 1\,.$$

In particular:

Restrictions among the Oseledets directions

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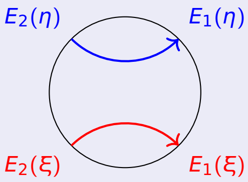
In particular:

*co-parallel configuration is **forbidden** en $K^\top$:*



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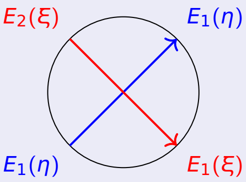

Consequences of the Geometrical Lemma

Consider the set $G := \left\{ (E_1(\xi), E_2(\xi));\ \xi \in K^\top \right\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

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Since the co-parallel configuration is forbidden, the set G has a **monotonicity property**: if E_1 moves clockwise then so does E_2.

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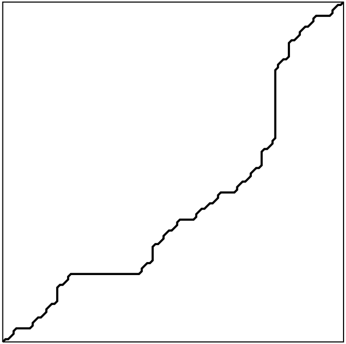
Since the co-parallel configuration is forbidden, the set G has a **monotonicity property**: if E_1 moves clockwise then so does E_2.

This implies that G is a graph (above its projection), with the exception to an at most countable numbers of "plateaux" and "cliffs":



Geometrical Lemma ⇒ Main Proposition

Recall the Main Proposition

Proposition

$NOC^+ \Rightarrow E_1$ determines E_2 almost surely with respect to any λ_1-maximizing or -minimizing measure.

Let μ be an ergodic non-periodic (and so non-atomic) measure supported in $K^\top$. (The periodic case is trivial.)

Then the (countable) union of plateaux and cliffs in G is the image under (E_1, E_2) of a set of zero μ-measure. (Here we need the NOC^+.)

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The case of $K^\perp$ is similar (but extra care is needed: monotonicity is only local).

Summary

As explained before, we obtain zero entropy since **the past** ξ_- **uniquely the future** ξ_+ μ**-a.e.**:

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Barabanov functions

$\Downarrow$

Geometrical Lemma (cross-ratios, forbidden configurations)

$\Downarrow$

Main Proposition

THE END (?)

PROOF OF THE GEOMETRICAL LEMMA

Some important ideas in the proof come from the 2002 paper by Bousch and Mairesse (counter-examples to the finiteness conjecture).

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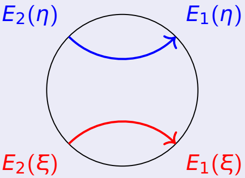

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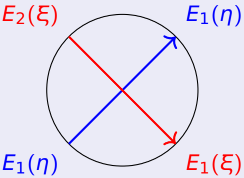
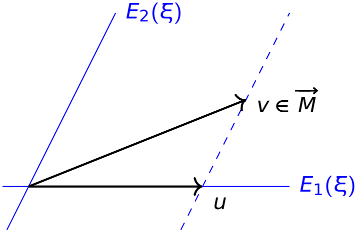


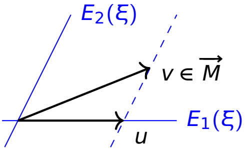
An important estimate

Lemma

Let $\xi \in K^\top$, $u \in E_1(\xi)$. Let $v \in \overrightarrow{M}$ be such that $u - v \in E_2(\xi)$. Then $|||u||| \le |||v|||$.



Proof of the “important estimate” Lemma



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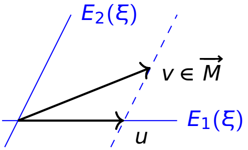

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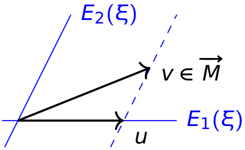

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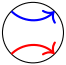
To prove the Lemma we show that $\boxed{Q_n \to 0}$ as $n \to +\infty$.

That’s easy, using that that $\angle(u_n, v_n) \to 0$ exponentially fast and “Lipschitz property” of the Barabanov norms.

Proof of the Geometrical Lemma

Assume for a contradiction that $\xi, \eta \in K^\top$ are in co-parallel configuration: .

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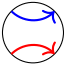
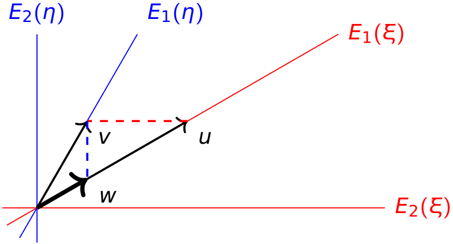
Assume for a contradiction that ξ, $\eta \in K^\top$ are in co-parallel configuration: . Choose a nonzero vector $u \in E_1(\xi)$, and let v, w be as in the figure.



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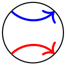
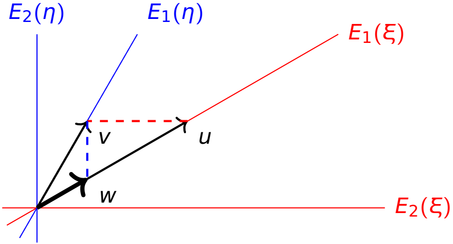
Assume for a contradiction that ξ, $\eta \in K^\top$ are in co-parallel configuration: . Choose a nonzero vector $u \in E_1(\xi)$, and let v, w be as in the figure.



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Apply the “important estimate” twice:

$$|||u||| \le |||v||| \le |||w|||. \quad \text{Contradiction!}$$