Title: Invariant measures for stochastic Navier-Stokes equations in unbounded domains via bw-Feller property

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Abstract

In this talk I will describe a general result on the existence of invariant measure for Markov processes having the bw-Feller property and will show how this can be applied to stochastic Navier-Stokes equations in unbounded domains. This talk is based on joint works with M Ondrejat and Ela Motyl. The results presented are in some sense generalisations of related results for stochastic nonlinear beam and wave equations obtained in a joint work with M. Ondrejat and J. Seidler.

Invariant measures

What are the classical methods of proving the existence of an invariant measure for a Markov proceess?

(1) An "invariant measure" is known (e.g. guessed or given) and one has to prove existence of a corresponding Markov process

(2) Krylov-Bogoliubov method for a given Markov proceess. This requires existence of an auxiliary set which is compactly embedded into the space and in which the Marov process is bounded in probability (uniformly for large times).

(3) Avez method.

(4) via an existence of a random compact invariant set

In SPDEs method (1) is used very much since Kryloff & Bogoliouboff (1937) and Oxtoby & Ulam (1939) for dynamical systems of compact (later locally compact) state spaces.

For instance for gradient systems with the so called white noise.

Method (2) has been used for e.g. Navier-Stokes equations.

Method (3) have been introduced by Avez (1968) and used by e.g. Lasota & Pianigianni (1977)

However, as mentioned above it requires existence of some compact embedding and thus is works for bounded domains only and equation with smoothing (e.g. with drift containing an analytic semigroup).

Method (4) As a byproduct of results obtained by myself and Yuhong Li (TAMS, 2006) about the existence of a random attractor is that there are invariant measure for stochastic 2-dimensional Navier-Stokes equations with additive noise in unbounded domains. Behind the proof is the continuity of the corresponding flow with respect to the **weak topologies**.

Coincidentally, Maslowski and Seidler (Atti Naz. Lincei 1999) proposed to use the of weak topologies to the proof of the existence of invariant measures. These two papers have inspired us to investigate this matter further.

It had become apparent while working on the existence of solutions to geometric wave equations that our fine techniques used to overcome the difficulty arising from having only weak a'priori estimates should also allow one to prove the sequentially weak Feller property required by the Maslowski and Seidler approach.

Our approach allows not only to prove the existence of an invariant measure for stochastic 2-d Navier-Stokes equations with multiplicative noise in unbounded domains and thus generalizing the previously mentioned my result with Y Li, but also consider stochastic beam and wave equations (joint work with M. Ondrejat and J. Seidler).

bw-Feller property

H a separable Hilbert space,

H_w the space X endowed with the weak topology

$C_b(H, \mathbb{R})$ e bounded real valued continuous functions on H

$C_b(H_w)$ bounded real valued continuous functions on H_w

$SC_b(H_w)$ bounded real valued sequentially continuous functions on H_w

$$P : \mathbb{R}_+ \times H \times \mathscr{B}(H) \to \mathbb{R}$$

a time-homogeneous transition probability in H such that for every $\Gamma \in \mathscr{B}(H)$, the function, $P(\cdot, \cdot, \Gamma) : \mathbb{R}_+ \times H \to \mathbb{R}$ is Borel.

The corresponding semigroup on $C_b(H, \mathbb{R})$ is denoted by (P_t), the adjoint semigroup is denoted by (P_t^*), i.e.

$$(P_t f)(x) := \int_H f(y) P(t, x, dy); \quad (t, x) \in \mathbb{R}_+ \times H.$$

Let the semigroup (P_t) be **sequentially weakly Feller**, i.e.

$$P_t(C_b(H_w) \subset SC_b(H_w), \;\; t > 0.$$

Proposition 1.1. *(Maslowski and Seidler (1999)) If also there exist a a Borel probability measure ν on H (e.g. $\nu = \delta_a$, $a \in H$) and $T_0 \geq 0$: $\forall \varepsilon > 0 \ \ \exists R > 0$*

$$\sup_{t \geq T_0} \frac{1}{t} \int_0^t \left(P_s^* \nu\right) \left(X \setminus B_R\right) ds \leq \varepsilon, \tag{1.1}$$

then there exists an invariant probability μ measure for U, i.e. satisfying

$$P_t^*(\mu) = \mu, \ \ t > 0.$$

The bounded-weak topology.

A set $A \subset H$ is closed in the **bw**-topology iff for every $R > 0$,

$$A \cap B_R \text{ is w-closed in } B_R.$$

where $B_R = \{x \in H : |x| \leq R\}$.

Note: w-topology $\tau_w \subset \tau_{bw}$ bw-topology

Notation: H_{bw} the space H endowed with the bw-topology

$C_b(H_{bw})$ bounded real valued continuous functions on H_{bw}

Proposition 1.2. ***Properties of bw-topogy.*** *For a function $f : H \to \mathbb{R}$ the following conditions are equivalent.*

(i) $f : H_{bw} \to \mathbb{R}$ *is continuous,*

(ii) for every $R > 0$, $f_{|B_R} : B_R \to \mathbb{R}$ *is weakly continuous,*

(iii) $f : H_w \to \mathbb{R}$ *is sequentially continuous (i.e.* $f : H \to \mathbb{R}$ *is sequentially weakly continuous).*

Definition A transition semigroup (P_t) is bw-Feller, iff

$$P_t(C_b(H_{bw}) \subset C_b(H_{bw}), \ \ t > 0.$$

Remark If (P_t) is bw-Feller, then it sequentially weakly Feller.

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2-D Navier-Stokes equations

- Let us begin with the classical formulation of the Navier-Stokes equations. Let $\mathcal{O} \subset \mathbb{R}^d$, where $d = 2, 3$, be an open and connected subset of $\mathbb{R}^d$ with regular boundary. We consider flow of an incompressible fluid filling the domain $\mathcal{O}$, described by the following partial differential equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\, \Delta u + \nabla p = f \quad \text{in} \quad (0,T) \times \mathcal{O}$$

with the incompressibility condition

$$\operatorname{div} u = 0 \quad \text{in} \quad (0,T) \times \mathcal{O},$$

the homogeneous Dirichlet boundary condition $u_{|\partial\mathcal{O}} = 0$ in $(0,\infty) \times \partial\mathcal{O}$ and the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathcal{O}.$$

Given:

- $\nu > 0$ - the kinematic viscosity, (we assume that $\nu := 1$),
- $f : [0,\infty) \times \mathcal{O} \to \mathbb{R}^d$ - external forces,
- $u_0 : \mathcal{O} \to \mathbb{R}^d$ - the initial velocity of the fluid.

Unknown:

- $u : [0,\infty) \times \mathcal{O} \to \mathbb{R}^d$ - the velocity of the fluid,
- $p : [0,\infty) \times \mathcal{O} \to \mathbb{R}$ - the pressure of the fluid.

Let us consider the space $\mathcal{V}$ of **divergence-free** test vector fields:

$$\mathcal{V} := \{\phi \in \mathrm{C}_c^\infty(\mathcal{O}, \mathbb{R}^d) : \quad \operatorname{div} \phi = 0\}.$$

If a pair $(u, p) \in \mathrm{C}^2([0, T] \times \bar{\mathcal{O}}; \mathbb{R}^d) \times \mathrm{C}^1([0, T] \times \bar{\mathcal{O}}; \mathbb{R})$ is a classical solutions to the Navier-Stokes problem, then it is also a weak one, i.e.

$$\begin{aligned}
\int_{\mathcal{O}} u(t,x) \cdot \phi(x)\, dx \quad &+ \quad \int_0^t \int_{\mathcal{O}} \left[(u(s) \cdot \nabla) u(s)\right] \cdot \phi \, dx ds \\
&+ \quad \int_0^t \int_{\mathcal{O}} \nabla u(s) \cdot \nabla \phi \, dx ds \\
&= \quad \int_{\mathcal{O}} u(0,x) \cdot \phi(x)\, dx + \int_0^t \int_{\mathcal{O}} f(s) \cdot \phi \, dx ds.
\end{aligned}$$

This can be proved by multiplying the NSEs by $\phi \in \mathcal{V}$, integrating over $\mathcal{O}$ and using the Stokes formula.

- Put also

$$\text{H} := \text{the closure of } \mathcal{V} \text{ in } L^2(\mathcal{O}, \mathbb{R}^d),$$
$$\text{V} := \text{the closure of } \mathcal{V} \text{ in } H^1(\mathcal{O}, \mathbb{R}^d).$$

H is endowed with the scalar product and norm inherited from $L^2(\mathcal{O}, \mathbb{R}^d)$

In V we consider the scalar product and norm inherited from the Sobolev space $H^1(\mathcal{O}, \mathbb{R}^d)$, i.e.

$$(u|v)_{\text{V}} := (u|v)_{L^2} + ((u|v)) = (u|v)_{L^2} + (\nabla u|\nabla v)_{L^2}$$

$$|u|^2_{\text{V}} := |u|^2_H + \|u\|^2 = |u|^2_H + |\nabla u|^2_{L^2}.$$

- The operator $\mathcal{A}$. Let

$$\mathcal{A}u := ((u|\cdot)), \qquad u \in V.$$

Notice that if $u \in V$, then $\mathcal{A}u \in \mathrm{V}'$ and $|\mathcal{A}u|_{\mathrm{V}'} \leq |u|$.

- The bilinear operator B. Let us consider the following tri-linear form

$$b(u, w, v) = \int_{\mathcal{O}} (u \cdot \nabla w) v \, dx = \sum_{i=1}^{d} \int_{\mathcal{O}} u_i \frac{\partial w}{\partial x_i} \cdot v \, dx.$$

The form b satisfies the following inequality

$$|b(u, w, v)| \leq c|u|_V |w|_V |v|_V, \qquad u, w, v \in V.$$

Moreover, if we define a bilinear map B by $B(u,w) := b(u,w,\cdot)$, then $B(u,w) \in V'$ for all $u,w \in V$ and that the following inequality holds

$$|B(u,w)|_{\mathrm{V}'} \le c|u|_V |w|_V, \qquad u,w \in V.$$

Thus the map $B : V \times V \to \mathrm{V}'$ is bilinear and continuous. Moreover,

$$\langle B(u,w)|v\rangle = -\langle B(u,v)|w\rangle, \qquad u,w,v \in V$$

and, in particular,

$$\langle B(u,v)|v\rangle = 0, \qquad u,v \in V.$$

Stochastic 2-D Navier-Stokes equations in unbounded domains

- Let $\mathcal{O} \subset \mathbb{R}^d$ be an open connected, unbounded subset with smooth boundary $\partial\mathcal{O}$, where $d = 2, 3$. We will consider the stochastic Navier-Stokes equations

$$\begin{aligned} du(t) &+ \big[(u \cdot \nabla)u - \Delta u + \nabla p\big]\, dt \\ &= f(t)\, dt + G(t, u(t))\, dW(t), \qquad t \in [0, T], \text{ in } \mathcal{O}, \\ \operatorname{div} u &= 0, \end{aligned}$$

with the initial condition

$$u(0) = u_0,$$

and with the homogeneous boundary condition $u_{|\partial\mathcal{O}} = 0$.

- In this problem

- $u = u(t,x) = (u_1(t,x), ...u_d(t,x))$ and $p = p(t,x)$ represent the velocity and the pressure of the fluid, respectively,

- f stands for the deterministic external forces. The terms

- $G(t,u(t))\, dW(t)$, where W is a cylindrical Wiener process on some separable Hilbert space K, stand for the random forces.

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, i.e. for every $t \geq 0$, $\mathcal{F}_t \subset \mathcal{F}$ and

$$\mathcal{F}_s \subset \mathcal{F}_t, \quad \text{if } s \leq t.$$

Abstract formulation of the problem

- The stochastic Navier-Stokes equation can be written as the following stochastic integral evolution equation

$$u(t) + \int_0^t [\mathcal{A}u(s) + B(u(s))]\, dt = u_0 + \int_0^t f(s)\, dt + \int_0^t G(t, u(s))\, dW(s), \quad t \in [0, \infty),$$

- Let $(\mathcal{O}_R)_{R\in\mathbb{N}}$ be a sequence of open and bounded subsets of $\mathcal{O}$ with regular boundaries $\partial\mathcal{O}_R$ such that

$$\mathcal{O}_R \subset \mathcal{O}_{R+1} \quad \text{and} \quad \bigcup_{R=1}^{\infty} \mathcal{O}_R = \mathcal{O}.$$

Let us define the following space

$L^2(0,T;H_{loc})$:= the space of measurable functions $u:[0,T]\to H$ such that for all $R\in\mathbb{N}$

$$p_{T,R}(u) := \Big(\int_0^T \int_{\mathcal{O}_R} |u(t,x)|^2 dx dt\Big)^{\frac{1}{2}} < \infty,$$

with the topology generated by the seminorms $(p_{T,R})_{R\in\mathbb{N}}$.

Assumptions. We assume that

(H.1) $W = (W(t))_{t\geq 0}$ is a cylindrical Wiener process in a separable Hilbert space K defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a complete filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$.

(H.2) A measurable map $G : H \to \mathcal{T}_2(\mathrm{K}, \mathrm{V}')$ such that

(ii) there exists $C > 0$ such that

$$\|G(u)\|^2_{\mathcal{T}_2(\mathrm{K},V')} \leq C(1 + |u|^2_H), \qquad u \in H. \tag{G2}$$

(i) $G(\mathrm{v}) \in \mathcal{T}_2(\mathrm{K}, H)$, for $\mathrm{v} \in V$, the map $G : V \to \mathcal{T}_2(\mathrm{K}, H)$ is locally Lipschitz and, for some constants λ_0, ρ and $\eta \in (0, 2]$,

$$2\langle \mathcal{A}u|u\rangle - \|G(u)\|^2_{\mathcal{T}_2(\mathrm{K},H)} \geq \eta\|u\|^2 - \lambda_0|u|^2_H - \rho, \qquad u \in \mathrm{V}, \tag{G1}$$

(iii) and, for every $\psi \in \mathcal{V}$ the function

$$\psi^{**}G : H_{\rm loc} \ni u \mapsto \Big\{ \mathrm{K} \ni y \mapsto {}_{\mathrm{V}'}\langle G(u)y, \psi \rangle_{\mathrm{V}} \in \mathbb{R} \Big\} \in \mathrm{K}' \quad \text{(G2')}$$

is continuous.

(H.3) Let

$$p \in [2, 2 + \frac{\eta}{2-\eta}), \tag{1.2}$$

$f \in L^p_{loc}([0,\infty); \mathrm{V}')$ and

$u_0 : \Omega \to H$ is an $\mathcal{F}_0$-measurable random variable such that

$$\mathbb{E}|u_0|^p_H < \infty.$$

Fix $s > \frac{d}{2}$ and define a Hilbert space

$$\mathrm{V}_s := \text{the closure of } \mathcal{V} \text{ in } H^{s,2}(\mathcal{O}, \mathbb{R}^d).$$

Choose a separable Hilbert space U such that U is a dense in V_s and

$$\text{the natural embedding } \iota_s : U \hookrightarrow \mathrm{V}_s \text{ is compact.} \tag{1.3}$$

Then we also have

$$U \hookrightarrow \mathrm{V}_s \hookrightarrow H \cong H' \hookrightarrow \mathrm{V}_s' \hookrightarrow U', \tag{1.4}$$

Consider, for fixed $T > 0$,

$$
\begin{array}{rcl}
\mathrm{C}([0,T],U') & & \text{with norm} \\
& & |u|_{\mathrm{C}([0,T],U')} := \sup_{t\in[0,T]} |u(t)|_{U'} \\
L^2_w(0,T;\mathrm{V}) & := & \text{the space } L^2(0,T;\mathrm{V}) \text{ with the weak topology,} \\
L^2(0,T;H_{\mathrm{loc}}) & & \text{defined before} \\
\mathrm{H}_w & & \text{the space H endowed with the weak topology} \\
\mathrm{C}([0,T];H_w) & := & \Big\{ u:[0,T]\to \mathrm{H}_w \text{ continuous } \Big\} \\
& & \text{topology generated by family of maps with } h\in H, \\
& & \mathrm{C}([0,T];H_w) \ni u \mapsto \big(u(\cdot)|h\big)_H \in \mathrm{C}([0,T];\mathbb{R}).
\end{array}
$$

Definition 1.3. For $T > 0$ let us put

$$\mathcal{Z}_T := \mathrm{C}([0,T];U') \cap L^2_w(0,T;\mathrm{V}) \cap L^2(0,T;H_{\mathrm{loc}}) \cap \mathrm{C}([0,T];H_w)$$

$\mathcal{T}_T$ is the supremum of the corresponding four topologies.
On $\mathcal{Z}_T$ we consider the Borel σ-algebra.

Lemma 1.4. *The sets* $\mathrm{C}([0,T];H) \cap \mathcal{Z}_T$, $\mathrm{C}([0,T];\mathrm{V}) \cap \mathcal{Z}_T$ *and* $L^2(0,T;V) \cap \mathcal{Z}_T$ *are Borel subset* $\mathcal{Z}_T$ *and the corresponding embeddings map the Borel sets into Borel subsets of* $\mathcal{Z}_T$*. Moreover, the following* $\mathbb{R}_+ \cup \{+\infty\}$*-valued functions are Borel*

$$\mathcal{Z}_T \ni u \mapsto \begin{cases} \sup_{s\in[0,T]} |u(s)|^2_H, & \textit{if } u \in \mathrm{C}([0,T];H) \cap \mathcal{Z}_T \\ \infty, & \textit{otherwise}, \end{cases}$$

$$\mathcal{Z}_T \ni u \mapsto \begin{cases} \int_0^T \|u(s)\|^2\, ds, & \textit{if } u \in L^2(0,T;V) \cap \mathcal{Z}_T, \\ \infty & \textit{otherwise}. \end{cases}$$

Definition 1.5. Let μ_0 be a Borel probability measure on H. We say that there exists **a martingale** solution of equation (1.2) on the interval $[0,T]$ with the initial distribution μ_0 iff there exist

a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ with a complete filtration $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_t\}_{t\geq 0}$,

a K-cylindrical Wiener process $\hat{W} = (\hat{W})_{t\geq 0}$.

an $\hat{\mathbb{F}}$-progressively measurable process $u : [0,T] \times \hat{\Omega} \to H$ such that

$$u(\cdot,\omega) \in \mathrm{C}\big([0,T], H_w\big) \cap L^2(0,T;V) \ \hat{\mathbb{P}}\text{-a.e.} \tag{1.5}$$

and the law of $u(0)$ on H is equal to μ_0

for all $t \in [0,T]$ and all $v \in \mathcal{V}$, $\hat{\mathbb{P}}$-a.s.,

$$\begin{aligned}
&\big(u(t)|v\big)_H + \int_0^t \langle \mathcal{A}u(s)|v\rangle\, ds + \int_0^t \langle B(u(s),u(s))|v\rangle\, ds \\
&= \big(u(0)|v\big)_H + \int_0^t \langle f(s)|v\rangle\, ds + \Big\langle \int_0^t G(u(s))\, d\hat{W}(s), v\Big\rangle
\end{aligned} \tag{1.6}$$

and

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |u(t)|_H^2 + \int_0^T |\nabla u(t)|^2\, dt\Big] < \infty. \tag{1.7}$$

If all the above conditions are satisfied, the system

$$(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$$

will be called a martingale solution to problem (1.2) on the interval $[0,T]$ with the initial distribution μ_0.

In the case when μ_0 is equal to the law on H of a given random variable $u_0 : \Omega \to H$ then, it is called a martingale solution to problem (1.2) on $[0, T]$ with the initial data having the same law as u_0. In particular, in this case we require that the laws on H of u_0 and $u(0)$ are equal.

When $T = \infty$ we can define a martingale solution to problem (1.2) on time interval $[0, \infty)$.

The existence of a solution

Theorem 1.6. *Under the above assumptions, there exists a martingale solution* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$ *on* $[0, \infty)$ *to the problem (1.2) such that for every* $T > 0$*,*

$$\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]} |\bar{u}(t)|_H^p + \int_0^T |\bar{u}(t)|_V^2 \, dt\Big] < \infty.$$

In the 2-dimensional case, if for some $C > 0$ *and* $L < 2$*,*

$$|G(u_1) - G(u_2)|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \le C|u_1 - u_2|^2_{\mathrm{H}} + L|u_1 - u_2|^2_{\mathrm{V}}, \quad u_1, u_2 \in \mathrm{V},$$

the solution is unique in law.
Moreover, if $\mathcal{O}$ *is a Poincaré domain, i.e.* $\exists\, C > 0$*:*

$$C \int_{\mathcal{O}} \varphi^2 \, d\xi \le \int_{\mathcal{O}} |\nabla \varphi|^2 \, d\xi \quad \text{for all } \varphi \in H_0^1(\mathcal{O}) \tag{1.8}$$

and inequality (G1) *holds with* $\lambda_0 = 0$*, then there exists a martingale solution* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, u)$ *of problem* (1.2) *s.t. for every* $t > 0$

$$\mathbb{E}\,|u(t)|_H^2 + \frac{\eta}{2}\mathbb{E}\int_0^t |\nabla u(s)|^2\,ds \leq \mathbb{E}\,|u_0|_H^2 + \frac{2}{\eta}\int_0^t |f(s)|_{\mathrm{V}'}^2\,ds + \rho t. \quad (1.9)$$

- The proof of the Theorem is based on the Faedo-Galerkin approximation, tightness and Jakubowski-Skorokhod Theorem

- G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

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Step 1. For each $n \in \mathbb{N}$, there exists a unique $\mathbb{F}$-adapted, continuous H_n-valued process u_n satisfying the Galerkin approximating equation.

- Using the Itô formula and maximal inequalities, we can prove the following lemma about estimates of the solutions u_n of the Galerkin equation.

Lemma 1.7. *The processes $(u_n)_{n\in\mathbb{N}}$ satisfy the following estimates.*

(i) For every $p \in [1,\infty)$ there exists a positive constant $C_1(p)$ such that

$$\sup_{n\geq 1} \mathbb{E}\big(\sup_{0\leq s\leq T} |u_n(s)|_H^p \big) \leq C_1(p).$$

(ii) There exists a positive constant C_2 such that

$$\sup_{n\geq 1} \mathbb{E}\Big[\int_0^T |u_n(s)|_V^2 \, ds \Big] \leq C_2.$$

Step 2. **(tightness criterion)** *Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}$-adapted U'-valued processes such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\,\big[\sup_{s\in[0,T]} |X_n(s)|_H^2\big] < \infty, \tag{1.10}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T |X_n(s)|^2\,ds\right] < \infty, \tag{1.11}$$

(a) *and for every $\varepsilon > 0$ and for every $\eta > 0$ there exists $\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}$-stopping times one has*

$$\sup_{n\in\mathbb{N}} \sup_{0\le\theta\le\delta} \mathbb{P}\{\, |X_n(\tau_n+\theta) - X_n(\tau_n)|_{U'} \ge \eta \,\} \le \varepsilon. \tag{1.12}$$

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on the Borel σ-field $\mathcal{B}(\mathcal{Z}_T)$. Then for every $\varepsilon > 0$ there exists a compact subset K_ε of $\mathcal{Z}_T$ such that

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \ge 1-\varepsilon.$$

Arzela-Ascoli Theorem in the space $C([0,T],H_w)$

H_w is not metrizable

$C([0,T],H_w)$ locally convex topological space with seminorms

$$|h|_\varphi := \sup\{|\langle \varphi, h(t)\rangle : t \in [0,T]\}$$

Put

$$\delta_\varepsilon(f) = \sup\{|f(a) - f(b)| : \, a, b \in [0,T], \, |a-b| \le \varepsilon\}.$$

Proposition 1.8. *Assume that* $\{\phi_k\}_{k\in\mathbb{N}} \subseteq X^*$ *separates points of* X*. If* $\alpha > 0$, $\beta = (\beta_n^k)_{k,n\in\mathbb{N}}$, $\lim_{n\to\infty} \beta_n^k = 0$ *for every* $k \in \mathbb{N}$*, then the set*

$$\begin{aligned} K^{\alpha,\beta,\phi} \quad := \quad & \Big\{ h \in C([0,T]; H_w) : \|h\|_{L^\infty(0,T;X)} \le \alpha, \\ & \delta_{\frac{1}{n}}(\phi_k \circ h) \le \beta_n^k, \ \text{for all } k, n \in \mathbb{N} \Big\} \end{aligned}$$

is compact in $C([0,T]; H_w)$*.*
Conversely, if $K \subset C([0,T]; H_w)$ *is compact then there exist* α*,* β *as above such that*

$$K \subseteq K^{\alpha,\beta,\phi}.$$

Definition 1.9. $\mathscr{B}_t(C([0,T];H_w))$ the σ-field on $C([0,T];H_w)$ generated by

$$C([0,T];H_w) \to H_w : h \mapsto h(s) \in H_w, \quad s \in [0,t]$$

Let $\{\varphi_n\}$ be dense in B_1. For $q \in \mathbb{Q} \cap [0,T]$, put

$$\xi_{n,q} : C([0,T];H_w) \ni h \mapsto \langle \varphi, h(q) \rangle \in \mathbb{R}.$$

Proposition 1.10. *The family* $\{\xi_{n,q}\}$ *separates points on* $C([0,T];H_w)$ *and*

$$\sigma\big(\{\xi_{n,q}\}\big) = \mathscr{B}_T(C([0,T];H_w))$$

Step 3. **The Jakubowski-Skorokhod Embedding Theorems**

Theorem 1.11 (Jakubowski, 1998). *Let* $(\mathcal{X}, \tau)$ *be a topological space such that there exists a sequence* (f_m) *of continuous functions* $f_m : \mathcal{X} \to \mathbb{R}$ *that separates points of* $\mathcal{X}$*. Let* (X_n) *be a sequence of* $\mathcal{X}$ *valued random variables. Suppose that for every* $\varepsilon > 0$ *there exists a compact subset* $K_\varepsilon \subset \mathcal{X}$ *such that*

$$\sup_{n \in \mathbb{N}} \mathbb{P}(\{X_n \in K_\varepsilon\}) > 1 - \varepsilon.$$

Then there exists a subsequence $(X_{n_k})_{k \in \mathbb{N}}$*, a sequence* $(Y_k)_{k \in \mathbb{N}}$ *of* $\mathcal{X}$ *valued random variables and an* $\mathcal{X}$ *valued random variable* Y *defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

$$\mathcal{L}(X_{n_k}) = \mathcal{L}(Y_k), \qquad k = 1, 2, ...$$

and for all $\omega \in \Omega$*:*

$$Y_k(\omega) \xrightarrow{\tau} Y(\omega) \quad \text{as } k \to \infty.$$

- Z. Brzeźniak, M. Ondrejat, *Stochastic geometric wave equations with values in compact Riemannian homogeneous spaces*, Ann. Probab. 41, no. 3B, 1938-1977 (2013)

- A. Jakubowski, *The almost sure Skorohod representation for subsequences in nonmetric spaces*, Teor. Veroyatnost. i Primenen. **42**, no. 1, 209-216 (1997); translation in Theory Probab. Appl. **42** no.1, 167-174 (1998).

Step 4. The limit process is a solution.

Step 5. Pathwise uniqueness for $d = 2$. This implies uniqueness in law.

Observations

The above proof can be extended to give the following result about the continuous dependence of solutions on the parameters (in the case $d = 2$).

Theorem 1.12. *Let* $d = 2$ *and the assumtions of Theorem 1.6 be satisfied. Assume that* $u_0 \in H$, $f \in \mathrm{V}'$ *and that an* H*-valued sequence* $(u_{0,n})_{n=1}^{\infty}$ *is weakly convergent in* H *to* u_0, *and that an* V'*-valued sequence* $(f_n)_{n=1}$ *is weakly convergent in* V' *to* f. *Let*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u_n)$$

be a strong solution of problem (1.2) *on* $[0, \infty)$ *with the initial data* $u_{0,n}$ *and the external force* f_n. *Then there exist a subsequence* $(n_k)_k$ *and*

- *a stochastic basis* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, *where* $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0}$,
- *a* K *cylindrical Wiener process* $\tilde{W} = (\tilde{W}(t))_{t \in [0,\infty)}$
- *and progressively measurable processes* $\tilde{u}(t)$, $(\tilde{u}_{n_k}(t))_{k \geq 1}, t \in [0, T]$ *(defined on this basis) with laws supported in* $\mathcal{Z}_T$ *such that*

$$\tilde{u}_{n_k} \textit{ has the same law as } u_{n_k} \textit{ on } \mathcal{Z}_T \textit{ and } \tilde{u}_{n_k} \to \tilde{u} \textit{ in } \mathcal{Z}_T, \quad \tilde{\mathbb{P}} \text{ - a.s.}$$

and the system $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u})$ *is a solution to problem* (1.2) *on the interval* $[0, \infty)$ *with the initial law* δ_x*. In particular, for all* $t \geq 0$, $\mathrm{v} \in \mathrm{V}$

$$\begin{aligned}(\tilde{u}(t)|\mathrm{v})_H - (\tilde{u}(0)|\mathrm{v})_H + \int_0^t \langle \mathcal{A}\tilde{u}(s)|\mathrm{v}\rangle\, ds + \int_0^t \langle B(\tilde{u}(s))|\mathrm{v}\rangle\, ds \\ = \int_0^t \langle f|\mathrm{v}\rangle\, ds + \Big\langle \int_0^t G(\tilde{u}(s))\, d\tilde{W}(s), \mathrm{v}\Big\rangle.\end{aligned}$$

Moreover, the process $\tilde{u}$ *satisfies the following inequality with* p *satisfying condition* (1.2) *and*

$$\mathbb{E}\,\Big[\sup_{s\in[0,T]} |\tilde{u}(s)|_H^p\Big] + \mathbb{E}\Big[\int_0^T \|\tilde{u}(s)\|^2\, ds\Big] < \infty. \qquad (1.13)$$

bw-Feller property for the 2-d Stochastic NSEs

For any bounded Borel function $\varphi \in \mathcal{B}_b(\mathrm{H})$ and $t \geq 0$ we define

$$(P_t\varphi)(x) = \mathbb{E}[\varphi(u(t,x))], \quad x \in \mathrm{H}. \tag{1.14}$$

Since the trajectories $u(\cdot, x)$ are continuous, $(P_t)_{t\geq 0}$ is a stochastically continuous semigroup on the Banach space $\mathcal{C}_b(\mathrm{H})$. This means that for every $\varphi \in \mathcal{C}_b(\mathrm{H})$ and $x \in \mathrm{H}$

$$\lim_{t\to 0} P_t\varphi(x) = x.$$

The following result is rather standard.

Proposition 1.13. *The family $u(t,x)$, $t \geq 0$, $x \in \mathrm{H}$ is Markov. In particular, $P_{t+s} = P_t P_s$ for $t, s \geq 0$.*

Proposition 1.14. *Then the semigroup P_t is bw-Feller, i.e. if $\phi : \mathrm{H} \to \mathbb{R}$ is a bounded sequentially weakly continuous function and $t > 0$ then $P_t\phi : \mathrm{H} \to \mathbb{R}$ is also a bounded sequentially weakly continuous function, i.e. if $x_n \rightharpoonup x$ weakly in H then*

$$P_t\phi(x_n) \to P_t\phi(x).$$

Proof of Proposition 1.14. Fix $t > 0$, $x \in \mathrm{H}$, an H-valued sequence $(x_n) \rightharpoonup x$ weakly in H and a function

$\phi : H \to \mathbb{R}$ is a bounded sequentially weakly continuous

Since the function $P_t\phi : H \to \mathbb{R}$ is bounded, we only need to prove that it is sequentially weakly continuous.

Let $u_n(\cdot) = u(\cdot, x_n)$ be a strong solution of problem (1.2) on $[0, \infty)$ with the initial law δ_{x_n}

Let $u(\cdot) = u(\cdot, x)$ be a strong solution of problem (1.2) on $[0, \infty)$ with the initial law δ_x

We assume that these processes are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$. Recall that

$$\mathcal{Z}_t := \mathrm{C}([0,t];U') \cap L^2_w(0,t;\mathrm{V}) \cap L^2(0,t;H_{\mathrm{loc}}) \cap \mathrm{C}([0,t];H_w).$$

By Theorem 1.12 about the continuous dependence there exist

- a subsequence $(n_k)_k$,
- a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_s\}_{s \in [0,t]}$,
- a cylindrical Wiener process $\tilde{W} = \tilde{W}(s)$, $s \in [0,t]$ defined on this basis,

- and progressively measurable processes $\tilde{u}(s)$, $\left(\tilde{u}_{n_k}(s)\right)_{k\geq 1}, s \in [0,t]$ (defined on this basis) with laws supported in $\mathcal{Z}_t$, such that

$$\tilde{u}_{n_k} \textit{ has the same law as } u_{n_k} \textit{ on } \mathcal{Z}_t \textit{ and } \tilde{u}_{n_k} \to \tilde{u} \textit{ in } \mathcal{Z}_t, \quad \tilde{\mathbb{P}} \textit{ - a.s.}$$

and the system $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u})$ which is a martingale solution to problem (1.2) on the interval $[0,t]$ with the initial law δ_x.

In particular, by the definition of $\mathcal{Z}_t$, $\tilde{\mathbb{P}}$-almost surely

$$\tilde{u}_{n_k}(t) \to \tilde{u}(t) \text{ weakly in } \mathrm{H}.$$

Since $\phi : H \to \mathbb{R}$ is sequentially weakly continuous, $\tilde{\mathbb{P}}$-a.s.,

$$\phi(\tilde{u}_{n_k}(t)) \to \phi(\tilde{u}(t)) \text{ in } \mathbb{R}.$$

Thus, as $\phi : H \to \mathbb{R}$ is bounded,

$$\lim_{k\to\infty} \tilde{\mathbb{E}}[\phi(\tilde{u}_{n_k}(t))] = \tilde{\mathbb{E}}[\phi(\tilde{u}(t))]. \tag{1.15}$$

Since the laws of $\tilde{u}_{n_k}$ and u_{n_k}, on $\mathcal{Z}_t$ are equal,

$$\tilde{\mathbb{E}}[\phi(\tilde{u}_{n_k}(t))] = \mathbb{E}[\phi(u_{n_k}(t))] = P_t\phi(x_{n_k}). \tag{1.16}$$

But both u and $\tilde{u}$ are solutions to (1.2) with the initial law δ_x, by uniqueness in law, their laws on $\mathcal{Z}_t$ are equal. In particular,

$$\tilde{\mathbb{E}}[\phi(\tilde{u}(t))] = \mathbb{E}[\phi(u(t))] = P_t\phi(x). \tag{1.17}$$

Thus by (1.15), (1.16) and (1.17), we infer that

$$\lim_{k\to\infty} P_t\phi(x_{n_k}) = P_t\phi(x).$$

By the sub-subsequence argument, the whole sequence $(P_t\phi(x_n))_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} P_t\phi(x_n) = P_t\phi(x). \qquad \text{QED}$$

□

Existence of an invariant measure

From inequality (1.9) and the Poincaré inequality (1.8), it follows that the following inequality holds

$$\int_0^t \mathbb{E}|u(s)|_H^2\,ds \le \frac{2}{C\eta}|u_0|_H^2 + \frac{2}{C\eta}\Big(\frac{2}{\eta}|f|_{V'}^2 + \varrho\Big)t, \qquad t \ge 0. \quad (1.18)$$

Thus

Corollary 1.15. *Let $x \in \mathrm{H}$ and let $u(t)$, $t \ge 0$, be the unique solution to the problem (1.2) starting from x. Then there exists $T_0 \ge 0$ such that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\sup_{T \ge T_0} \frac{1}{T} \int_0^T (P_s^* \delta_x)(H \setminus \bar{B}_R)\,ds \le \varepsilon, \quad (1.19)$$

where $\bar{B}_R = \{\mathrm{v} \in H : \ |\mathrm{v}|_H \le R\}$.

Theorem 1.16. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a Poincaré domain. Let assumptions (H.1)-(H.2) be satisfied. In addition we assume that*

$$|G(u_1) - G(u_2)|^2_{\mathcal{T}_2(Y,H)} \leq C|u_1 - u_2|^2_H + L|u_1 - u_2|^2, \quad u_1, u_2 \in V.$$

for some $C > 0$ and $L < 2$, and that inequality (G1) *holds with $\lambda_0 = 0$. Then there exists an invariant measure of the semigroup $(P_t)_{t\geq 0}$ defined by* (1.14), *i.e. a probability measure μ on H such that*

$$P_t^* \mu = \mu.$$

Open Problem

Is the invariant measure μ from the last Theorem unique?

Deterministic example

We will present now the earlier promised example based on the paper Brzeźniak and Li (2006).

Example 1.17. If $\varphi = (\varphi_t)_{t \geq 0}$ is a deterministic dynamical system on a Hilbert space H, then one can define the corresponding Markov semigroup by

$$[P_t(f)](x) := f(\varphi_t(x)), \;\; t \geq 0, \;\; x \in \mathrm{H}. \tag{1.20}$$

Suppose that the semiflow is sequentially weakly continuous in the following sense.

$$\text{If } t_n \to t \in \mathbb{R}_+, \;\; x_n \rightharpoonup x \text{ in } \mathrm{H} \text{ then } \varphi_{t_n}(x_n) \rightharpoonup \varphi_t(x) \text{ in } \mathrm{H}. \tag{1.21}$$

Note that the above condition is satisfied for the deterministic 2-d Navier-Stokes equations, see Rosa (1998) and also Lemma 7.2 in B. & Li (2006).

Then, for any bounded sequentially weakly continuous function $f : \mathrm{H} \to \mathbb{R}$, any sequences (t_n) and (x_n) are such that $t_n \to t$ in $\mathbb{R}_+$ and $x_n \to x$ weakly in H, the following holds

$$P_{t_n} f(x_n) \to P_t f(x), \tag{1.22}$$

i.e. the semigroup (P_t) is sequentially weakly Feller.

Indeed, let us choose and fix a bounded sequentially weakly continuous function $f : \mathrm{H} \to \mathbb{R}$, a sequence $(t_n) \to t$ and a sequence (x_n) such that $x_n \to x$ weakly in H. Then by assumption (1.21) $\varphi_{t_n}(x_n) \to \varphi_t(x)$ weakly in H and since f is sequentially weakly continuous we infer that

$$[P_{t_n}(f)](x_n) = f(\varphi_{t_n}(x_n)) \to f(\varphi_t(x)) = P_t f(x).$$

The condition (1.1) from the Maslowski-Seidler Theorem now reads: There exists $x \in \mathrm{H}$ such for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t 1_{[R,\infty)}(|\varphi_s(x)|_{\mathrm{H}})\, ds \le \varepsilon \tag{1.23}$$

which is obviously satisfied provided the dynamical system $\varphi = (\varphi_t)_{t\ge 0}$ is bounded at infinity, i.e. there exists $x \in \mathrm{H}$ and $R > 0$ such that $|\varphi_s(x)|_{\mathrm{H}} \le R$ for all $s \ge 0$. It is well known that this condition holds for the deterministic 2-d Navier-Stokes equations in a Poincaré domain (as well as for the damped Navier-Stokes Equations in the whole space $\mathbb{R}^2$.

Thus we conclude, that in those cases, there exists an invariant measure. Of course, these are known results, the purpose of this Example is only to elucidate our paper by showing that it is also applicable to these cases.

Let us point out that Lemma 7.2 in my paper with Li (2006) played an important rôle in that paper.

I believe that the result described in this Example holds also for the Random dynamical system generated by 2-D stochastic Navier-Stokes Equations in unbounded domain. In this way, we will get an alternative proof of the result existence of an invariant measure proved in that paper.