Synchronization by noise

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Outline

Introduction

Introduction

Synchronization by noise

- We consider SDE on $\mathbb{R}^d$ of the type

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \tag{*}$$

- The inclusion of noise may simplify the long-time dynamics, i.e. while

$$dX_t = b(X_t)dt$$

  may not be globally stable, the long-time behavior of $(*)$ may be trivial.
- Roughly speaking: Synchronization by noise means that the random attractor consists of a single random point, i.e.

$$A(\omega) = \{a(\omega)\}, \quad \mathbb{P}\text{-a.s.}$$

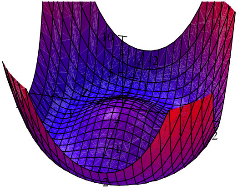
- In particular: If synchronization occurs, then each two trajectories converge to each other in probability:

$$|X_t^x - X_t^y| \to 0 \quad \text{for } t \to \infty$$

  in probability.

Model example

- Double-well potential, $V(x) = -\frac{1}{2}|x|^2 + \frac{1}{4}|x|^4$



with additive Wiener noise, i.e.

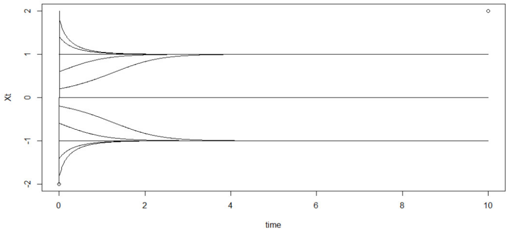
$$dX_t = (X_t - |X_t|^2 X_t)dt + \sigma dW_t.$$

Model example

- Deterministic case ($\sigma = 0,\ d = 1$):

$$dX_t = (X_t - X_t^3)dt$$

- Attractor is given by closed unit ball: $A = \bar{B}_1(0) = [-1,1]$.
- Point attractor is given by $S^{d-1} \cup \{0\} = \{\pm 1, 0\}$.
- Simulation:

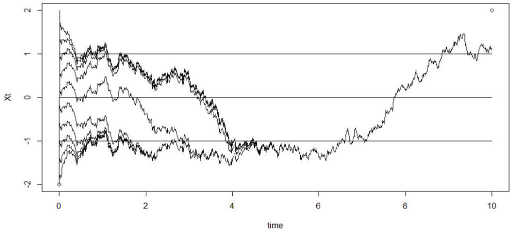


Model example

- Additive noise ($\sigma > 0$):

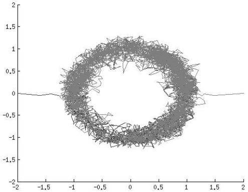
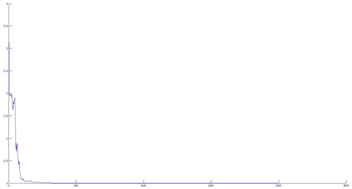
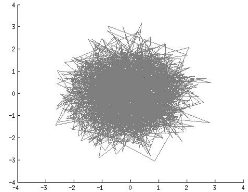
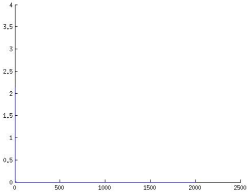
$$dX_t = (X_t - X_t^3)dt + \sigma dW_t$$

- Synchronization occurs: $A(\omega) = \{a(\omega)\}$ a.s.. In particular $|X_t^x - X_t^y| \to 0$ for $t \to \infty$ in probability.
- Simulation:



Model example

- Starting point of presented work: How to prove this for $d > 1$?

	Trajectories	Distance of trajectories
$\sigma = 1$		
$\sigma = 10$		

Known methods

There are several distinct methods to prove synchronization by noise available in the literature (there are many more!):

1. Order-preserving RDS + uniqueness of invariant measure (e.g. Arnold, Chueshov '98; Chueshov, Scheutzow '04)
    - For $d = 1$ this proves synchronization for
    $$dX_t = (X_t - X_t^3)dt + \sigma dW_t, \quad \sigma > 0.$$
    - Problem: No preserved partial order on $\mathbb{R}^d$ for $d > 1$
2. Local stability + transitivity of the two-point motion (e.g. Baxendale '91)
    - Transitivity of the two-point motion completely unclear for additive noise
3. Perturbation techniques/large deviation methods (e.g. Tearne '08, Martinelli, Scoppola, '88, '94)
    - Essential assumption: The drift b has only finitely many fixed points.
4. ... (many more, e.g. Master-slave synchronization (Chueshov, Schmalfuss '10)) ...

Model example

Question

Open question in the literature: Does synchronization occur for

$$dX_t = (X_t - |X_t|^2 X_t)dt + \sigma dW_t$$

with $\sigma > 0$ *and* $d > 1$*?*

A new approach to synchronization

A new approach to synchronization

General setup

In the following let φ be a white noise RDS, (E,d) be a Polish space.

Definition

A *weak random attractor* is a random compact set $A(\omega)$ such that

1. (invariance): $\varphi_t(\omega)A(\omega) = A(\theta_t\omega)$, a.s. for all $t \geq 0$.
2. (attraction):

$$d(\varphi_t(\omega)B, A(\theta_t\omega)) \to 0 \quad \text{for } t \to \infty$$

   in probability, for each compact set B.

If we replace compact sets B by points, then A is called a *weak point attractor*.

Definition

We say that *synchronization* occurs if the weak random attractor is a singleton

$$A(\omega) = \{a(\omega)\} \quad \text{a.s..}$$

We say that *weak synchronization* occurs if there is a singleton weak point attractor.

Local stability

Definition

Let $U \subset E$ be a (deterministic) non-empty open set. We say that φ is *asymptotically stable* on U if there exists a (deterministic) sequence $t_n \uparrow \infty$ such that

$$\mathbb{P}\big(\lim_{n\to\infty} \mathrm{diam}(\varphi_{t_n}(.,U)) = 0\big) > 0.$$

- We will see later that a negative top Lyapunov exponent implies asymptotic stability.
- In the following assume that φ has a weak random attractor A.

Lemma

Let φ be asymptotically stable on U and assume

$$\mathbb{P}(A \subset U) > 0.$$

Then A is a singleton $\mathbb{P}$-a.s., i.e. synchronization holds.

Full support for the attractor

Definition

We say that φ is *swift transitive* if, for every closed ball $B(x,r)$ and every point y, there is a time $t>0$ such that

$$\mathbb{P}(\varphi_t(\cdot,B(x,r))\subset B(y,2r))>0.$$

Lemma

If φ is swift transitive and

$$\operatorname{ess\,inf}\{\operatorname{diam}(A(\omega));\omega\in\Omega\}=0 \qquad (*)$$

then

$$\mathbb{P}(A\subset U)>0$$

for every non-empty (deterministic) open set $U\subset E$.

Condition $(*)$ means that $\mathbb{P}(\operatorname{diam}(A)<\varepsilon)>0$ for every $\varepsilon>0$.

Full support for the attractor

Theorem

Assume that φ is asymptotically stable on some non-empty open set $U \subset X$ and is swift transitive. Let A satisfy

$$\operatorname{ess\,inf}\{\operatorname{diam}(A(\omega)); \omega \in \Omega\} = 0 \qquad (*)$$

Then A is a singleton, i.e. synchronization occurs.

Small diameter

Definition

We say that φ is *contracting on large sets* if for every $R > 0$, there is a ball $B(y, R)$ and a time $t > 0$ such that

$$\mathbb{P}\left(\operatorname{diam}(\varphi_t(\cdot, B(y, R))) \leq \frac{R}{4}\right) > 0.$$

Lemma

Assume that φ is contracting on large sets and swift transitive. Then A has small diameter, i.e. (*) holds.

Examples

- How restrictive are the assumptions of asymptotic stability, swift transitivity and contraction on large sets?
- asymptotic stability:
  - Follows from local stable manifold theorem if $\lambda_{top} < 0$
  - For additive noise
    $$dX_t = b(X_t)dt + dW_t \qquad (*)$$
    we have the bound
    $$\lambda_{top} \leq \int_{\mathbb{R}^d} \lambda^+(x) d\mu(x),$$
    with $\lambda^+(x) := \max_{|v|=1}(Db(x)v, v)$.
  - For gradient systems, i.e. $b = -\nabla V$ and small noise one often has $\lambda_{top} < 0$. If $V(x) = g(|x|^2)$ with g convex, then always $\lambda_{top} < 0$.
- swift transitivity: Satisfied basically for all SDE with additive noise.
- contraction on large sets: Consider $(*)$ and assume that for all $R > 0$ there exists a ball $B(z, R)$ such that
  $$\langle b(x) - b(y), x - y \rangle < 0 \quad \forall x, y \in B(z, R).$$
  Then contraction on large sets holds.

Examples

In particular we obtain:

Example

Synchronization holds for

$$dX_t = (X_t - |X_t|^2 X_t)dt + \sigma dW_t$$

with $\sigma > 0$ and $d \geq 1$.

Weak synchronization

Weak synchronization

Weak synchronization

- What can we say without assuming eventual monotonicity?
- Let φ be a white noise RDS and assume that P_t is ergodic with invariant measure μ.
- A random probability measure $\omega \mapsto \mu_\omega$ is a measurable function from Ω to the space of probability measures. We say that μ_ω is φ-invariant if

$$\varphi_t(\omega)_* \mu_\omega = \mu_{\theta_t \omega} \quad \text{a.s.}.$$

Fact

If μ_ω is an $\mathscr{F}_0$-measurable random invariant measure, then $\mu = \mathbb{E}\mu_\omega$ is P_t-invariant. Conversely, if μ is P_t-invariant then

$$\mu_\omega = \lim_{t\to\infty} \varphi_t(\theta_{-t}\omega)_* \mu$$

exists for $\mathbb{P}$-a.e. ω, it is an $\mathscr{F}_0$-measurable random invariant measure.

Weak synchronization

Fact

Every $\mathscr{F}_0$-measurable random invariant measure is supported by the weak random point attractor, i.e.

$$\mu_\omega(A(\omega)) = 1 \quad a.s..$$

If φ is strongly mixing and $A(\omega) := supp(\mu_\omega)$ is compact then $A(\omega)$ is a (minimal) weak point attractor.

Lemma

The statistical equilibrium μ_ω is either discrete or diffuse. More precisely, either μ_ω consists of finitely many atoms of the same mass $\mathbb{P}$-a.s., i.e. there is an $N \in \mathbb{N}$ and $\mathscr{F}_0$-measurable random variables $a_1, \dots, a_N$ such that

$$\mu_\omega = \{\frac{1}{N}\delta_{a_i(\omega)} : i = 1, \dots, N\}$$

or μ_ω does not have point masses $\mathbb{P}$-a.s..

Weak synchronization

Local stability can now be nicely captured in terms of the structure of the statistical equilibrium, i.e.

Lemma

Assume that φ is weakly asymptotically stable on U with $\mu(U) > 0$, i.e. there exists a sequence $t_n \to \infty$ such that, for all $x, y \in U$

$$d(\varphi_{t_n}(\cdot, x), \varphi_{t_n}(\cdot, y)) \to 0$$

in probability. Then μ_ω is discrete.

Proposition

If φ is strongly mixing and weakly asymptotically stable on U with $\mu(U) > 0$, then there is an $N \in \mathbb{N}$ and $\mathscr{F}_0$-measurable random variables $a_1, \ldots, a_N$ such that

$$A(\omega) = \mathrm{supp}(\mu_\omega) = \{a_i(\omega) : i = 1, \ldots, N\}$$

is a minimal weak point attractor.

Weak synchronization

- It remains to show (under further assumptions) that trajectories get close. This replaces the assumption of eventual monotonicity/contraction on large sets.
- Let us consider gradient systems, i.e.

$$dX_t = -\nabla V(X_t)dt + \sigma dW_t$$

  and assume strong mixing, i.e. $\rho(x) := e^{-\frac{2}{\sigma^2}V(x)} \in L^1(\mathbb{R}^d)$.
- To prove that trajectories get close, we need some kind of monotonicity of $b = -\nabla V$. From $\rho(x) \in L^1(\mathbb{R}^d)$ we get: For all $s \in S^{d-1}$, $\delta > 0$ there is a $z \in \mathbb{R}^d$ such that

$$\langle b(z) - b(z - \delta s), s \rangle < 0.$$

Weak synchronization

Theorem

Assume that $\rho(x) := e^{-\frac{2}{\sigma^2}V(x)} \in L^1(\mathbb{R}^d)$ *and that* φ *is weakly asymptotically stable on* U *with* $\mu(U) > 0$*. Then, there is a minimal weak point attractor* A *consisting of a single random point* $a(\omega)$ *and*

$$A(\omega) = supp(\mu_\omega) = \{a(\omega)\} \quad \mathbb{P}\text{-a.s.},$$

i.e. weak synchronization holds.

Question

Open questions:

- *For gradient systems: Does weak asymptotic stability always hold?*
- *What about the Lorenz system?*

Order-preserving RDS

Order-preserving RDS
ongoing work with Franco Flandoli

Order-preserving RDS

- So far: Had to assume (weak) asymptotic stability. We show next: For order-preserving RDS this is unnecessary.
- Let (E,d) be a Polish space with closed partial order "$\leq$"
- Let φ be a white noise RDS and φ be strongly mixing with invariant measure μ, i.e. for all $x \in E$

$$\mathscr{L}(\varphi_t(\cdot,x)) \to \mu, \quad \text{for } t \to \infty.$$

- φ is order-preserving if for all $x \leq y$

$$\varphi_t(\omega,x) \leq \varphi_t(\omega,y) \quad \forall t \geq 0, \omega \in \Omega.$$

Order-preserving RDS

Theorem

Assume that μ is concentrated on intervals, i.e. for all $\varepsilon > 0$ there exists an interval $[f,g] \subseteq E$ such that

$$\mu([f,g]) \geq 1 - \varepsilon.$$

Then, the support of the statistical equilibrium is given by a single random point $supp\mu_\omega = \{a(\omega)\}$ and

$$A(\omega) := \{a(\omega)\}$$

is a singleton minimal weak point attractor, i.e. weak synchronization holds.

- Note: No asymptotic stability nor contraction on large sets required.

Order-preserving RDS

Key ingredient:

Proposition

Let φ be strongly mixing, order-preserving and $f \le g$. Then, for all $x, y \in [f, g]$:

$$d(\varphi_t(\omega, x), \varphi_t(\omega, y)) \to 0 \quad \text{for } t \to \infty$$

in probability. In other words: φ is weakly asymptotically stable on each interval $[f, g]$.

Order-preserving RDS and SPDE

- Generally speaking: Order-preservation corresponds to comparison principles for SPDE
- In the literature: To prove synchronization "compatibility" of "$\leq$" with the topology of E had to be assumed, i.e.
  - Admissibility: For each compact set $K \subseteq E$ there is an interval $[f,g] \subseteq E$ containing K.
  - Normality: $\mathrm{diam}([f,g]) \leq C\|f-g\|$ (in Banach spaces)
- Problem:
  - Admissibility is often false for SPDE, e.g. L^p spaces $p \in [1,\infty)$.
  - The following example shows that also normality can be too restrictive.

An application to SPDE

- We consider the stochastic porous medium equation

$$dX_t = \left(\Delta X_t^{[m]} + X_t\right) dt + dW_t,$$

  with zero Dirichlet boundary conditions on a bounded, smooth domain $\mathscr{O} \subseteq \mathbb{R}^d$, $d \leq 4$, $m > 1$.
- There is an associated RDS φ on $H^{-1} := (H_0^1)^*$ with strongly mixing invariant measure μ (assuming some non-degeneracy for W_t).
- For two distributions $x, y \in H^{-1}$ we can introduce the (standard) partial order "$\leq$" on H^{-1} by $x \leq y$ iff

$$(y - x)(v) \geq 0$$

  for all nonnegative $v \in H_0^1$.
- "$\leq$" is preserved by φ.
- We need to check that μ concentrates on intervals: For all $\varepsilon > 0$ exists a $[f, g]$ such that $\mu([f, g]) \geq 1 - \varepsilon$. **But**: We only know $\mu(L^{m+1}) = 1$.

An application to SPDE

- Key idea: Introduce alternative non-standard partial order: For $x, y \in H^{-1}$ we have $x \preceq y$ iff
$$(-\Delta)^{-1}x \leq (-\Delta)^{-1}y.$$
- Now:
  - "$\preceq$" is preserved by φ
  - μ is concentrated on intervals since $W^{2,m+1} \hookrightarrow L^\infty$.
- But:
  - "$\preceq$" is not normal: There are $f \preceq g$ such that $\mathrm{diam}([f,g]) = \infty$
  - "$\preceq$" is not admissible for $d > 1$.

Theorem

Weak synchronization holds for

$$dX_t = \left(\Delta X_t^{[m]} + X_t\right) dt + dW_t,$$

with $d \leq 4$, $m > 1$.

Thanks

Thanks!