Dynamics of reaction-diffusion equations with small Lévy noise

A. Debussche J. Gairing C. Hein M. Högele P. Imkeller I. Pavlyukevich

ENS Cachan, U de los Andes Bogota, HU Berlin, U Jena

Banff, January 20, 2015

Debussche, A., Högele, M., Imkeller, P.
The dynamics of nonlinear reaction-diffusion equations with small Lévy noise.
Lecture Notes in Mathematics 2085.
Springer: Berlin 2013.

1. Dynamical model: reaction-diffusion equation with Lévy noise

Example: EBM; energy balance: reaction term diffusion term: heat transport from equator to poles; noise: Lévy, includes Gaussian

Goal: study meta-stability properties of reaction-diffusion systems with α-stable Lévy noise of small intensity

state of the system in example: heat distribution $X^\varepsilon(t,\zeta)$: temperature at time t observed at $\zeta \in [0,1]$ (interval of global latitudes), subject to noise, intensity ε;

idealized version: reaction term $U(u) = \lambda(\frac{u^4}{4} - \frac{u^2}{2})$ $(\lambda > 0)$, diffusion operator $\frac{\partial^2}{\partial\zeta^2}$, noise: L Lévy process in $H = H_0^1([0,1])$

$$\begin{aligned} dX_t^\varepsilon(\zeta) &= [\frac{\partial^2}{\partial\zeta^2} X_t^\varepsilon(\zeta) - U'(X_t^\varepsilon(\zeta))]dt + \varepsilon dL_t(\zeta), \\ X_t^\varepsilon(0) &= X_t^\varepsilon(1) = 0, \quad t > 0, \\ X_0^\varepsilon(\zeta) &= x(\zeta), \quad \text{with } x \in H, \zeta \in [0,1]. \end{aligned}$$

2. Deterministic part: Chafee-Infante equation

For $\pi^2 < \lambda \neq (k\pi)^2, k \in \mathbf{N}$, consider

$$\begin{aligned} du_t &= \frac{\partial^2}{\partial\zeta^2} u_t - U'(u_t), && t \in [0,T], \\ u_t(0) &= u_t(1) = 0, && t \in [0,T], \\ u_0(\zeta) &= x(\zeta), && \zeta \in [0,1]. \end{aligned}$$

For initial values $x \in H_0^1([0,1])$ there is a unique continuous solution in $H = H_0^1([0,1])$, e.g. Chafee-Infante '74, Henry '83, Carr, Pego '89

There are two stable states $\{\varphi^+, \varphi^-\}$ with resp. domains of attraction D^+ and D^- and a smooth separatrix $\mathcal{S} = H \setminus (D^+ \cup D^-)$.

3. α-stable processes in Hilbert space $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$

$(L_t)_{t\geq 0}$ càdlàg version of symmetric α-stable process in H, Lévy measure

$$\nu(dy) = \sigma(ds)\frac{dr}{r^{1+\alpha}}, \quad r = \|y\|, \quad s = y/\|y\|, \quad y \neq 0,$$ σ finite measure on $\partial B_1(0)$

Lévy-Hinčin representation of characteristic function

$$\mathbb{E}\left[e^{i\langle u, L_t\rangle}\right] = \exp\left(t\int_H \left(e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbf{1}_{\{\|y\|\leq 1\}}\right)\nu(dy)\right), \quad u \in H.$$

for $\nu : \mathcal{B}(H) \to [0, \infty]$ satisfying $\int_H \left(1 \wedge \|y\|^2\right) \nu(dy) < \infty$,
$\nu(A) = \nu(-A), \quad A \in \mathcal{B}(H)$.

In contrast to (Q-)Brownian motion in H: no scalar decomposition

$$L_t \neq \sum_{k=1}^{\infty} \lambda_k L_t^k e_k$$ for ONB $(e_n) \subset H$, $\quad (L_t^k)_{t\geq 0}$ independent

L is heavy-tailed and need not have any moments.

4. Stochastic Chafee-Infante equation

For $\varepsilon > 0$ and $U'(y) = \lambda(y^3 - y)$

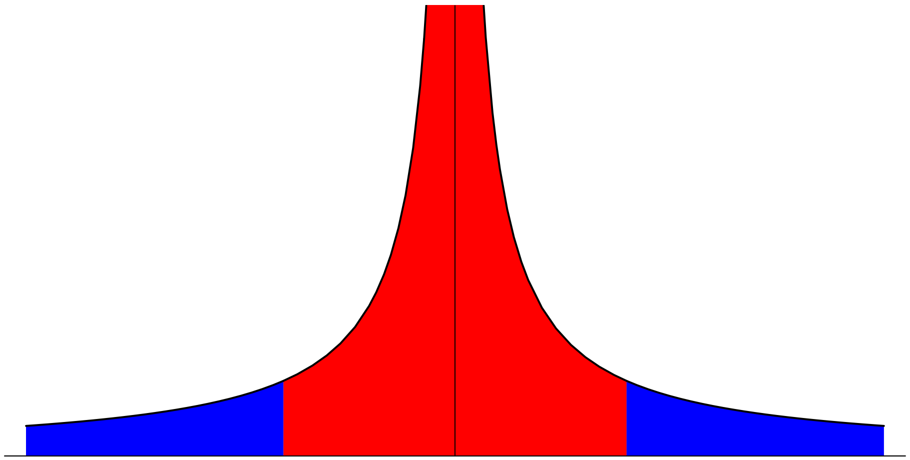
$$\begin{aligned} dX_t^\varepsilon(\zeta) &= [\frac{\partial^2}{\partial\zeta^2} X_t^\varepsilon(\zeta) - U'(X_t^\varepsilon(\zeta))]dt + \varepsilon dL_t(\zeta), \\ X_t^\varepsilon(0) &= X_t^\varepsilon(1) = 0, \quad t > 0, \\ X_0^\varepsilon(\zeta) &= x(\zeta), \quad \text{with } x \in H, \zeta \in [0,1]. \end{aligned}$$

There is a unique càdlàg mild solution in H (Peszat, Zabczyk '07, Marinelli, Röckner '09), satisfying the strong Markov property

$$X_t^\varepsilon = S(t)x - \int_0^t S(t-s) f(X_s^\varepsilon)\, ds + \varepsilon \int_0^t S(t-s)\, dL_s.$$

Pb: Let $\sigma_x^\varepsilon = \sigma^\varepsilon$ be exit time of X^ε starting in x from D^+. Describe the law of σ^ε asymptotically as $\varepsilon \to 0$.

5. Probabilistic approach of exit times



Decompose noise into small ($\leq \varepsilon^{-\frac{1}{2}}$) and large ($> \varepsilon^{-\frac{1}{2}}$) jumps.

$$L_t = \xi_t^\varepsilon + \eta_t^\varepsilon$$

$(\eta_t^\varepsilon)_{t\geq 0}$ compound Poisson process with jump measure ν^ε

$$\nu^\varepsilon(A) = \frac{d\nu\left(A \cap \varepsilon^{-\frac{1}{2}} B_1^c(0)\right)}{\beta_\varepsilon}, \quad A \in \mathcal{B}(H), \qquad \beta_\varepsilon = \nu\left(\varepsilon^{-\frac{1}{2}} B_1^c(0)\right) \sim \varepsilon^{\frac{\alpha}{2}}.$$

5. Probabilistic approach of exit times

$$\eta_t^\varepsilon = \sum_{T_k \le t} W_k, \quad \mathcal{L}(W_k) = \nu^\varepsilon$$

$$T_k = \sum_{i=1}^{k} \tau_i \qquad \mathcal{L}(\tau_i) = EXP(\beta_\varepsilon).$$

$(\xi_t^\varepsilon)_{t\ge 0}$ is a martingale with bounded jumps. Denote the stochastic convolution

$$\xi_t^* = \int_0^t S(t-s)\, d\xi_s^\varepsilon,$$

and the small jump part

$$Y_t^\varepsilon = S(t)x - \int_0^t S(t-s) f(Y_s^\varepsilon)\, ds + \varepsilon \xi_t^*.$$

6. Deviation in interjump intervals

Now

$$\int_0^{T_1} S(T_1 - s)\, dL_t = \int_0^{T_1-} S(T_1 - s)\, dL_s + \Delta_{T_1} L = \xi^*(T_1) + W_1.$$

Decompose

$$Y_t^\varepsilon(x) - u_t(x) = R_t^\varepsilon(x) + \varepsilon \xi_t^*.$$

Using properties of u prove that $|R_t^\varepsilon(x)|_\infty \to 0+$ if $\|\xi_t^*\| \to 0+$. Control small jumps convolution by

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|\varepsilon \xi_t^*\| \geq c\varepsilon^\gamma \right) \leq c_\xi T \varepsilon^{1-2\gamma+\frac{\alpha}{2}}.$$

Use this to estimate small jumps deviation: there is $\vartheta > \frac{\alpha}{2}, \gamma > 0$ such that

$$\mathbb{P}\left(\exists\, x \in D^+(\varepsilon^\gamma) : \sup_{t\in[0,T_1]} |Y_t^\varepsilon(x) - u_t(x)|_\infty \geq (1/2)\varepsilon^\gamma \right) \leq \varepsilon^\vartheta.$$

7. Relaxation in interjump intervals

Thm 1 (Temam '92):
All trajectories enter a large ball in $H_0^1([0,1])$ in uniform relaxation time T_{rec} .

Thm 2
For $\gamma \in (0,1)$ one can define a reduced domain of attraction $D^+(\varepsilon^\gamma), \varepsilon > 0,$ such that all trajectories starting in $D^+(\varepsilon^\gamma)$ enter $B(\varphi^+, \varepsilon^\gamma)$ before time $R_\varepsilon = c_\gamma |\ln \varepsilon|$.

one-dimensional case, $\varphi^+ = 0, x \in B_1(0)$: $-U'(u) \sim Mu$ for u near 0. Hence

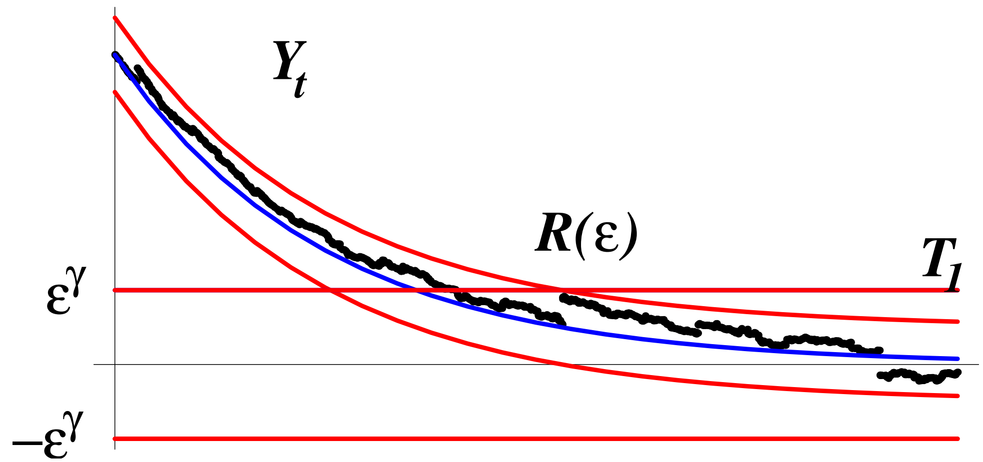
$$R_\varepsilon = \int_{\varepsilon^\gamma}^{x} \frac{1}{|U'(u)|} du \approx \int_{\varepsilon^\gamma}^{x} \frac{1}{Mu} du \approx \text{Const} + \frac{\gamma}{M} |\ln \varepsilon|.$$

8. Deviation/relaxation in interjump intervals

For small $\varepsilon > 0$

$$\mathbb{E}T_1 = 1/\beta_\varepsilon \sim \varepsilon^{-\frac{\alpha}{2}} > T_{rec} + c_\gamma |\ln \varepsilon|.$$

We can expect to make large jump from close to stable state!

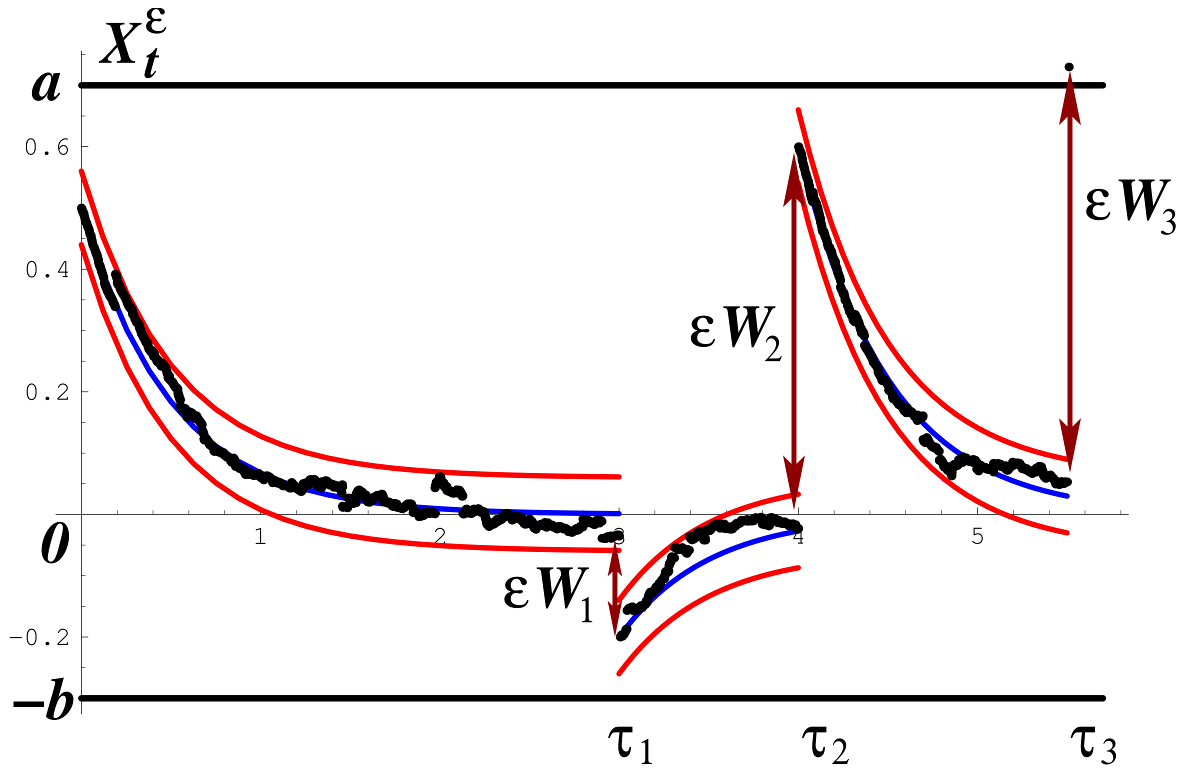


Thus large jumps $\varepsilon\eta^\varepsilon$ should dominate exit behavior for small $\varepsilon > 0$, exit times of polynomial order.

Denote $D_0^+(\varepsilon^\gamma) = D^+(\varepsilon^\gamma) - \varphi^+$ the shifted reduced domain. Consider the mass the reshifted domain of attraction

$$\lambda^+(\varepsilon) = \nu\left((1/\varepsilon)\left(D_0^+(\varepsilon^\gamma)\right)^c\right).$$

9. Typical behavior



Expected jump time	$\mathbf{E}\tau_k = \frac{1}{\beta_\varepsilon} = \frac{\alpha}{2}\varepsilon^{-\alpha/2}$	**polynomial in** ε
Relaxation time	$R_\varepsilon = \mathcal{O}(\lvert \ln \varepsilon \rvert)$	**logarithmic in** ε
Between big jumps	X^ε driven by	**“small jumps”** $\varepsilon\xi^\varepsilon$
Deviation probability		**small**

$\Rightarrow$ **Typically,** X^ε **jumps from a neighborhood of** 0 by εW_k at T_k.

Typically, X^ε **exits** $I = [-b, a]$ **by jumping at times** T_k.

10. Exit time law: heuristic proof

$$T_k = \tau_1 + \tau_2 + \cdots + \tau_k, \quad \mathbf{E}T_1 = 1/\beta_\varepsilon$$

$$\mathbf{P}(\varepsilon W_1 \notin [-b, a]) = \frac{1}{\beta_\varepsilon}\left(\int_{\frac{a}{\varepsilon}}^{\infty} + \int_{\frac{b}{\varepsilon}}^{\infty}\right)\frac{dy}{y^{1+\alpha}} = \frac{\varepsilon^\alpha}{\alpha\beta_\varepsilon}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]$$

$$\begin{aligned}
\mathbf{E}_x\sigma^\varepsilon &\approx \sum_{k=1}^{\infty} \mathbf{E}T_k \cdot \mathbf{P}(\sigma^\varepsilon = T_k) \\
&= \sum_{k=1}^{\infty} k \cdot \mathbf{E}T_1 \cdot \mathbf{P}(\varepsilon W_1 \in I, \ldots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I) \\
&= \sum_{k=1}^{\infty} k \cdot \mathbf{E}T_1 \cdot [1 - \mathbf{P}(\varepsilon W_1 \notin I)]^{k-1} \cdot \mathbf{P}(\varepsilon W_1 \notin I) \\
&= \frac{\mathbf{P}(\varepsilon W_1 \notin I)}{\beta_\varepsilon}\frac{1}{\mathbf{P}(\varepsilon W_1 \notin I)^2} = \frac{\alpha}{\varepsilon^\alpha}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]^{-1}
\end{aligned}$$

11. Asymptotic first exit time from a domain of attraction

Thm 3

For any $x \in D^+$ and $\theta > -1$

$$\lim_{\varepsilon \to 0+} \mathbb{E}_x \left[e^{-\theta \lambda^+(\varepsilon) \sigma^\varepsilon} \right] = \frac{1}{1+\theta} = \widehat{EXP(1)}.$$

Cor 1:

For all $x \in D^+$

$$\mathcal{L}\left(\lambda^+(\varepsilon)\sigma_x^\varepsilon\right) \longrightarrow EXP(1), \qquad \varepsilon \to 0+\,.$$

Moreover

$$\lim_{\varepsilon \to 0+} \lambda^+(\varepsilon)\mathbb{E}_x\left[\sigma^\varepsilon\right] = 1.$$

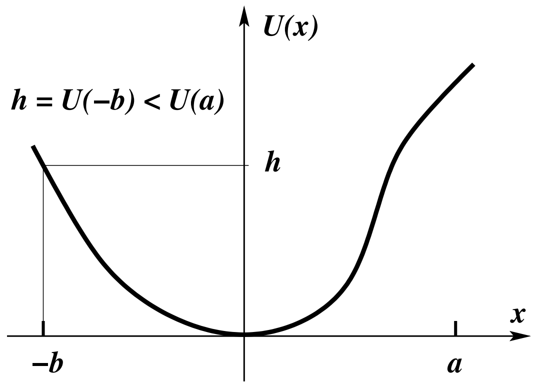
For small $\varepsilon > 0$

$$\mathbb{E}_x\left[\sigma^\varepsilon\right] = 1/\lambda^+(\varepsilon) \sim c/\varepsilon^\alpha.$$

Thus exit times have polynomial growth asymptotically!

12. comparison: Gaussian and Lévy dynamics, dim 1

$$\hat{\sigma}^\varepsilon = \inf\{t \geq 0 \,:\, \hat{X}^\varepsilon(t) \notin [-b, a]\}$$



$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(t))\, ds + \varepsilon W(t)$$

Thm 4 (Freidlin-Wentzell):

$$\mathbf{P}_x(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma}^\varepsilon < e^{(2h+\delta)/\varepsilon^2}) \to 1$$

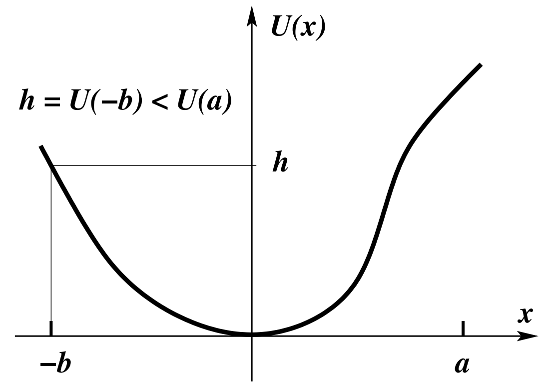
Kramers' law ('40, Williams, Bovier et al.):

$$\mathbf{E}_x \hat{\sigma}^\varepsilon \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\varepsilon^2}$$

Exponential law (Day, Bovier et al.)

$$\mathbf{P}_x\left(\frac{\hat{\sigma}^\varepsilon}{\mathbf{E}_x\hat{\sigma}^\varepsilon} > u\right) \sim \exp(-u)$$

$$\sigma^\varepsilon = \inf\{t \geq 0 \,:\, X^\varepsilon(t) \notin [-b, a]\}$$



$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\, ds + \varepsilon L(t)$$

Thm 5

$$\mathbf{P}_x\left(\frac{1}{\varepsilon^{\alpha-\delta}} < \sigma^\varepsilon < \frac{1}{\varepsilon^{\alpha+\delta}}\right) \to 1$$

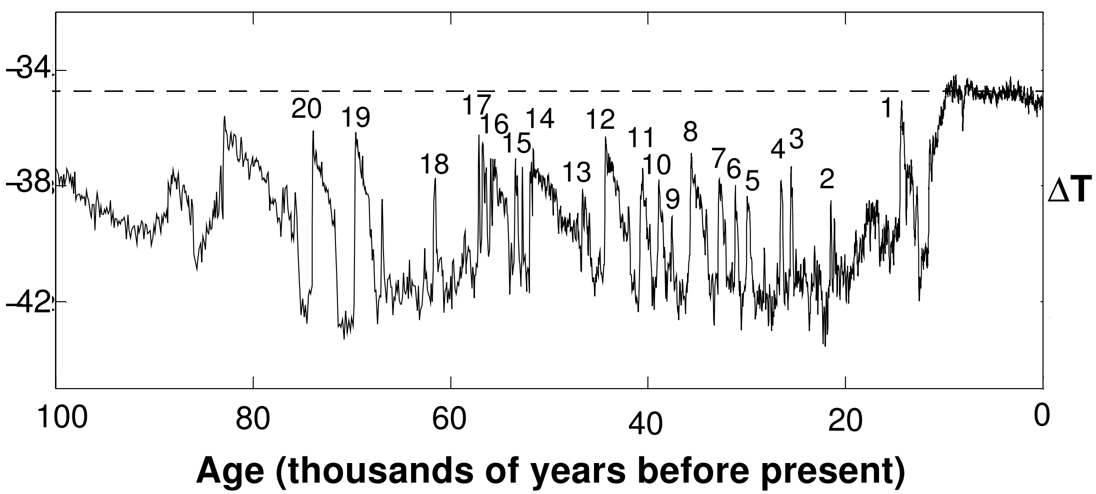
$$\mathbf{E}_x \sigma^\varepsilon \approx \frac{1}{\varepsilon^\alpha}\left(\int_{\mathbb{R}\setminus[-b,a]} \frac{dy}{|y|^{1+\alpha}}\right)^{-1}$$

$$\mathbf{P}_x\left(\frac{\sigma^\varepsilon}{\mathbf{E}_x\sigma^\varepsilon} > u\right) \sim \exp(-u)$$

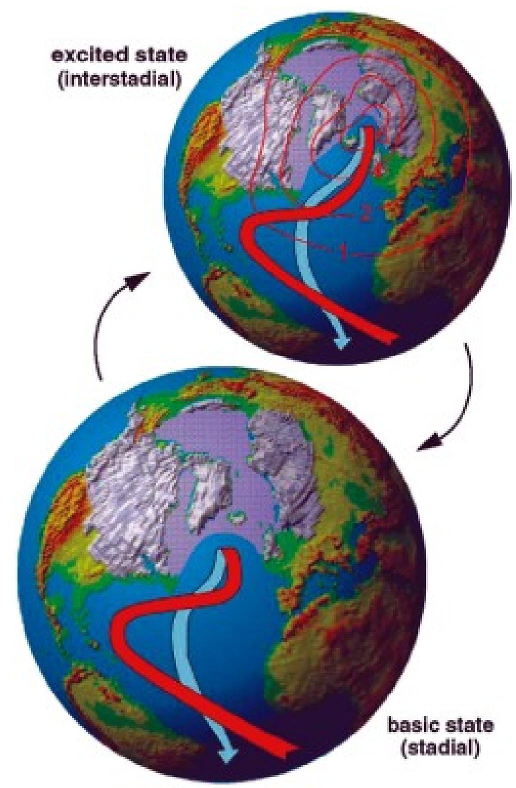
13. Paleo-climatic time series

temperature indicators: ^{18}O, ^{16}O, methane, calcium etc.

GRIP ice core data: 20 abrupt changes in climate of Greenland during last ice age (-91 000 to -11 000 y) (D/O events).



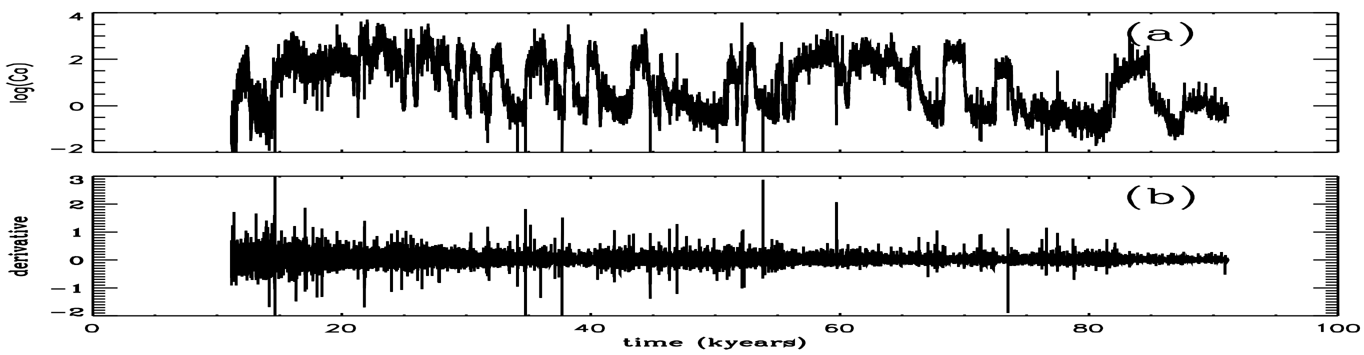
- rapid warming by 5-10°C within one decade
- subsequent slower cooling within a few centuries
- fast return to stable cold ground state



simulations: Ganopolsky/Rahmstorf, Potsdam Institute for Climate Impact Research

14. Dansgaard-Oeschger events. Statistical analysis

Calcium signal from GRIP: about 80 000 samples for 80 000 y



waiting times between D/O events: multiples of ∼1470 years.

What triggers the transitions?

modeling by Langevin equation *(*reaction):

$$dX(t) = -U'(t, X(t))dt + \text{NOISE}$$

U — multi well potential, depicts energy balance

P. Ditlevsen (*Geophys. Res. Lett. 1999*): power spectrum analysis of time series:
NOISE contains strong α-stable component with $\alpha \approx 1.75$.

15. p-Variation as test statistic

Which model of noise fits best with time series: estimate, test parameter

Ditlevsen's analysis: power spectrum of residua of time series

Problem: Stationarity?

Aim: better test statistics than peaks of power spectrum.

Model assumption: with some U interpret data as

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))ds + \varepsilon L(t) = Y^\varepsilon(t) + L^\varepsilon(t)$$

L Lévy process containing α-stable component with unknown α, Y^ε of bounded variation; estimate, test α

Idea: p-variation characteristic for fluctuation behavior of noise processes.

$$V_t^{p,n}(X) = \sum_{i=1}^{[nt]} |X(\frac{i}{n}) - X(\frac{i-1}{n})|^p, \quad V_t^p = \lim_{n\to\infty} V_t^{p,n}$$

16. α-stable Lévy Processes

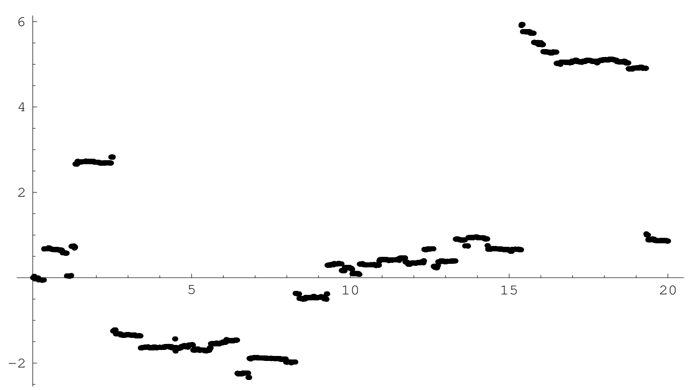
L Lévy process with characteristics (d, γ, ν) iff

$$E(e^{iuL(t)}) = e^{t(-\frac{1}{2}du^2 + i\gamma u + \int_{\mathbf{R}}[e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}]\nu(dy))}, \quad u \in \mathbf{R}, t \geq 0,$$

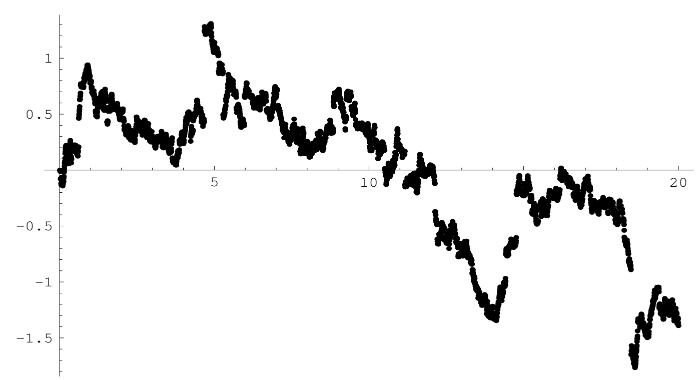
ν measure on Borel sets in $\mathbf{R}$ with $\nu(\{0\}) = 0$, $\int_{\mathbf{R}}[|y|^2 \wedge 1]\nu(dy) < \infty$.

L α-stable symmetric Lévy process if

$$E(e^{iuL(t)}) = e^{-c(\alpha)t|u|^\alpha}, \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}}dy, \quad u, y \in \mathbf{R}.$$



$\alpha = 0.75$



$\alpha = 1.75$

17. p-Variation and the Blumenthal-Getoor Index

L α-stable process with jump measure ν; then p-variation identified by

Blumenthal-Getoor index

$$\beta_L = \inf\{s \geq 0 : \int_{\{|y| \leq 1\}} |y|^s \nu(dy) < \infty\}$$

$$\gamma_L = \inf\{p > 0 : V_1^p(L) < \infty\}$$

Thm 1
L symmetric α-stable. Then

$$\gamma_L = \beta_L = \alpha.$$

Problem: How to read $\gamma_L = \alpha$ off the sequence $(V_t^{p,n}(L))_{n \in \mathbf{N}}$?

Calls for results about the asymptotic behavior of the sequence.

18. The case $\alpha = 2$: Brownian motion

For $n \in \mathbf{N}$ $V_1^{p,n}(W)$ consists of n independent increments and

$$E(V_1^{p,n}(W)) = n^{1-\frac{p}{2}} E(|W(1)|^p)$$

Thm 2 (LLN type)

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W) \to tE(|W(1)|^p) \quad \text{in probability,}$$

Y of bounded variation. Then also

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W+Y) \to tE(|W(1)|^p) \quad \text{in probability.}$$

Thm 3 (CLT type)

$$\begin{aligned}(n^{\frac{1}{2}}[n^{-1+\frac{p}{2}} V_t^{p,n}(W) - tE(|W(1)|^p)])_{t\geq 0} &= (n^{-\frac{1}{2}+\frac{p}{2}} V_t^{p,n}(W) - n^{\frac{1}{2}} tE(|W(1)|^p))_{t\geq 0} \\ &\to ((var(|W(1)|^p))^{\frac{1}{2}} \tilde{W}(t))_{t\geq 0}\end{aligned}$$

weakly with respect to the Skorokhod metric, and an independent Brownian motion $\tilde{W}$.

19. The case $\alpha < 2$

(Lit: Corcuera, Nualart, Wörner '07; case $p < \alpha$ for LLN type, $p < \frac{\alpha}{2}$ for CLT type)

Problem: $p < \frac{\alpha}{2} < 1$ not satisfactory for paleo-climatic data! Beyond $\frac{\alpha}{2}$ no CLT type result available, no asymptotic normality, but asymptotically of different type.

Thm 4 (LT type)
L α-stable with $\alpha \in]0, 2[$. Then

$$(V_t^{p,n}(L) - B_t^n(\alpha, p))_{t \geq 0} \to \tilde{L}$$

weakly with respect to the Skorokhod metric, and an independent $\frac{\alpha}{p}$-stable process $\tilde{L}$. Here

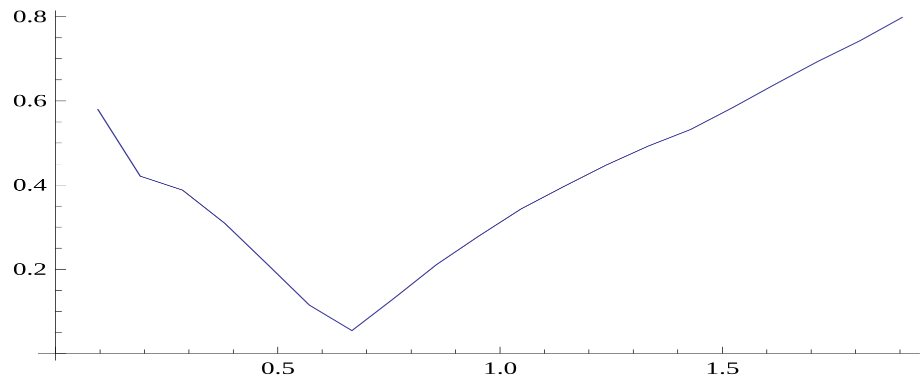
$$B_t^n(\alpha, p) = \begin{cases} n^{1-\frac{p}{\alpha}} t E(|L(1)|^p), & \frac{\alpha}{2} < p < \alpha, \\ n t^2 E(\sin((nt)^{-1} |L(1)|^p)), & p = \alpha, \\ 0, & \alpha < p. \end{cases}$$

Same result with $L + Y$ instead of L if Y is of finite p-variation and $\frac{\alpha}{2} < p < 1$ or $p > \alpha$.

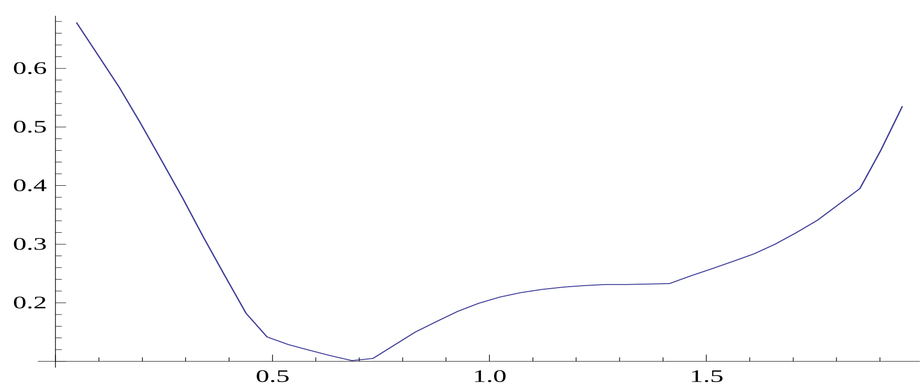
20. Test for α with real and simulated data

Thm 5: law of $V^{2\alpha,n}(X)$ converges to $\frac{1}{2}$-stable law if data of time series X have α-stable residuals

Kolmogorov-Smirnov statistics: distance between empirical law of $V^{2p,n}(X)$ and $\frac{1}{2}$-stable law, as a function of p; minimum of curve: right α



simulated time series of a 0.6-stable Levy process, $n = 200$



real time series from the Greenland ice, $n = 200$

21. Gaussian and Lévy dynamics: one dimension

W Wiener process

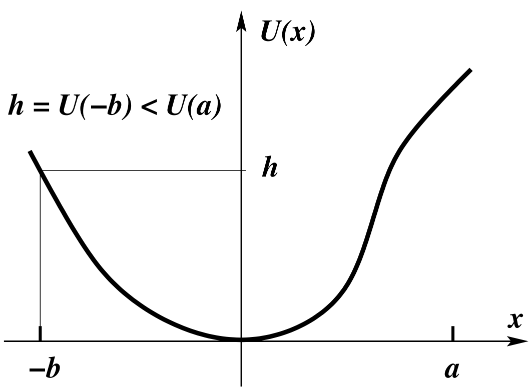
L symmetric α-stable Lévy process

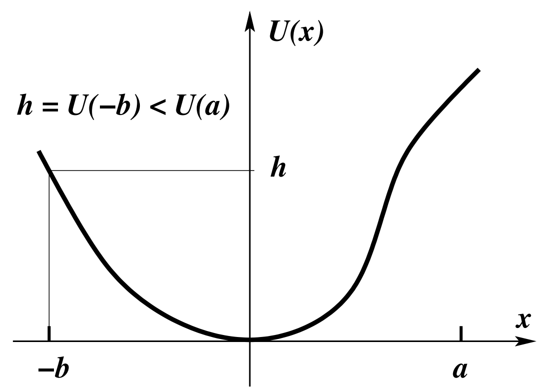
Tail behavior

$$P(|W(1)| \geq x) \sim \exp(-cx^2) \qquad P(|L(1)| \geq x) \sim c\frac{1}{x^\alpha},\ x \to \infty$$

$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(t))\, ds + \varepsilon W(t) \qquad X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\, ds + \varepsilon L(t)$$

$$\hat{\tau}(\varepsilon) = \inf\{t \geq 0\ :\ \hat{X}^\varepsilon(t) \notin [-b,a]\} \qquad \tau(\varepsilon) = \inf\{t \geq 0\ :\ X^\varepsilon(t) \notin [-b,a]\}$$





$$\mathbf{E}_x\hat{\tau}(\varepsilon) \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}}\exp\left(\frac{2h}{\varepsilon^2}\right) \qquad \mathbf{E}_x\tau(\varepsilon) \approx \frac{1}{\varepsilon^\alpha}\Big(\int_{\mathbb{R}\setminus[-b,a]}\frac{dy}{|y|^{1+\alpha}}\Big)^{-1}$$

22. Gaussian and Lévy dynamics: one dimension

Conjecture: make tails of Lévy process exponentially light to recover Gaussian exit behavior.

Tail behavior

L Lévy process with jump measure having tails

$$P(|L(1)| \geq x) \sim \exp(-cx^{\alpha}), \quad x \to \infty$$

sub-exponential tails: $\alpha < 1$ super-exponential tails: $\alpha > 1$

Consider

$$X^{\varepsilon}(t) = x - \int_0^t U'(X^{\varepsilon}(s-))\, ds + \varepsilon L(t)$$

$$\tau(\varepsilon) = \inf\{t \geq 0 \, : \, X^{\varepsilon}(t) \notin [-1,1]\}$$

Conjecture:

$$E_x(\tau(\varepsilon)) \sim_{\varepsilon \to 0} \exp(\frac{c}{\varepsilon^2}) \quad \text{as} \quad \alpha \uparrow 2.$$

23. The phase transition at $\alpha = 1$

Thm 10 [sub-exponential tails] For $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1-\delta)\exp(-C_\varepsilon^{1-\delta} t) \leq \sup_{|x|\leq 1} \mathbf{P}_x(\tau(\varepsilon) > t) \leq \exp(-\tfrac{1}{2} C_\varepsilon t),$$

with $C_\varepsilon := 2\nu([\frac{1}{\varepsilon}, \infty))$. Hence for $|x| < 1$

$$\lim_{\varepsilon \to 0} \varepsilon^\alpha \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

Thm 11 [super-exponential tails] q_ε ε-quantile of jump measure ν. Then for $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1-\delta)\exp(-D_\varepsilon^{1-\delta} t) \leq \sup_{|x|\leq 1} \mathbf{P}_x(\tau(\varepsilon) > t) \leq (1+\delta)\exp(-D_\varepsilon^{1+\delta} t),$$

where $D_\varepsilon = \exp(-d_\alpha \frac{|\ln \varepsilon|}{\varepsilon q_\varepsilon})$ and $d_\alpha = \alpha(\alpha-1)^{\frac{1}{\alpha}-1}$. Hence for $|x| < 1$

$$d_\alpha^{-1} \lim_{\varepsilon \to 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

24. α-stable towards Gaussian. The phase transition at $\alpha = 1$

Comparison of regimes for mean exit time

Power tails jump tails $\nu([x,\infty)) = x^{-r}$, $x \geq 1$ for some $r > 0$. Then

$$2 \lim_{\varepsilon\to 0} \varepsilon^r \mathbf{E}_x \tau(\varepsilon) = 1.$$

Sub-exponential tails jump tails $\nu([u,\infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha < 1$. Then

$$\lim_{\varepsilon\to 0} \varepsilon^\alpha \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

Super-exponential tails jump tails $\nu([u,\infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha > 1$. Then

$$d_\alpha^{-1} \lim_{\varepsilon\to 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

Gaussian diffusion no jumps, L one-dimensional Brownian motion. Then

$$\frac{1}{2}(U(-1) \wedge U(1))^{-1} \lim_{\varepsilon\to 0} \varepsilon^2 \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

25. Heuristics of exits: climbing versus jumping

The Brownian case

LD theory: diffusion has to climb potential in order to exit at lowest saddle point

The power tail case

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than $\frac{1}{\sqrt{\varepsilon}}$; small jump and Gaussian part ξ^ε with jumps not exceeding $\frac{1}{\sqrt{\varepsilon}}$; exit asymptotically due to one big jump, as shown in first talk

The case of exponential tails

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than g_ε; small jump and Gaussian part ξ^ε with jumps not exceeding this bound;

choose g_ε individually according to sub- and super-exponential tails

show that exit before time T while not returning to an interval around stable fixed point 0 of radius $\delta > 0$ requires that either increments of ξ^ε exceed certain bounds (for which probability is small enough), or sum of large jumps before time T exceeds bound $1 - \delta$

in any case large jumps responsible for exits

25. Heuristics of exits: climbing versus jumping

N_T random number of large jumps before time T

W_i jump $n^o\ i$, $i \in \mathbf{N}$.

N_T Poisson with expectation $\beta_\varepsilon T$, where $\beta_\varepsilon = \nu([-g_\varepsilon, g_\varepsilon]^c) = 2\exp(-x^\alpha)$

For n fixed, probability that sum of large jumps exceeds bound $1-\delta$ estimated by

$$P(N_T > n) + \sum_{k=1}^{n} P(N_T = k) P(\sum_{i=1}^{k} |\varepsilon W_i| > 1-\delta)$$

Idea for estimation:
for $n \in \mathbf{N}$

$$P(N_T > n) \leq (1+\delta)\exp(-n \ln n) \quad \text{(Stirling's formula)}$$

choose $n = n_\varepsilon$ suitably

25. Heuristics of exits: climbing versus jumping

essential term to estimate for $1 \leq k \leq n_\varepsilon$

$$P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta)$$

law of i.i.d. random variables $(|W_i|)_{i \in \mathbf{N}}$: $\beta_\varepsilon^{-1} 2\nu|_{[g_\varepsilon, \infty[}$

hence

$$\begin{aligned} P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta) &\leq \beta_\varepsilon^{-k} \exp(-\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i \geq \frac{1-\delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\}) \\ &= \beta_\varepsilon^{-k} \exp(-\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = \frac{1-\delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\}) \end{aligned}$$

minimization problem in the exponent of this estimate causes phase transition

24. Heuristics of exits: climbing versus jumping

By suitable choice of g_ε: lower boundary for x_i in $\inf$ can be taken 0.

sub-exponential tails

$$\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = 1, x_i \geq 0\} = 1$$

The minimum is taken on the boundary of the simplex, and $x_i = \frac{1}{n}, 1 \leq i \leq n,$ corresponds to maximum of the function

$$(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i^\alpha$$

25. Heuristics of exits: climbing versus jumping

Super-exponential tails

$$\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = 1, x_i \geq 0\} = n(\frac{1}{n})^\alpha$$

The minimum is taken for $x_i = \frac{1}{n}, 1 \leq i \leq n$, the unique local minimum of the function

$$(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i^\alpha$$

Bifurcation in the asymptotic behavior:
phase transition due to switch from concavity to convexity at $\alpha = 1$ of

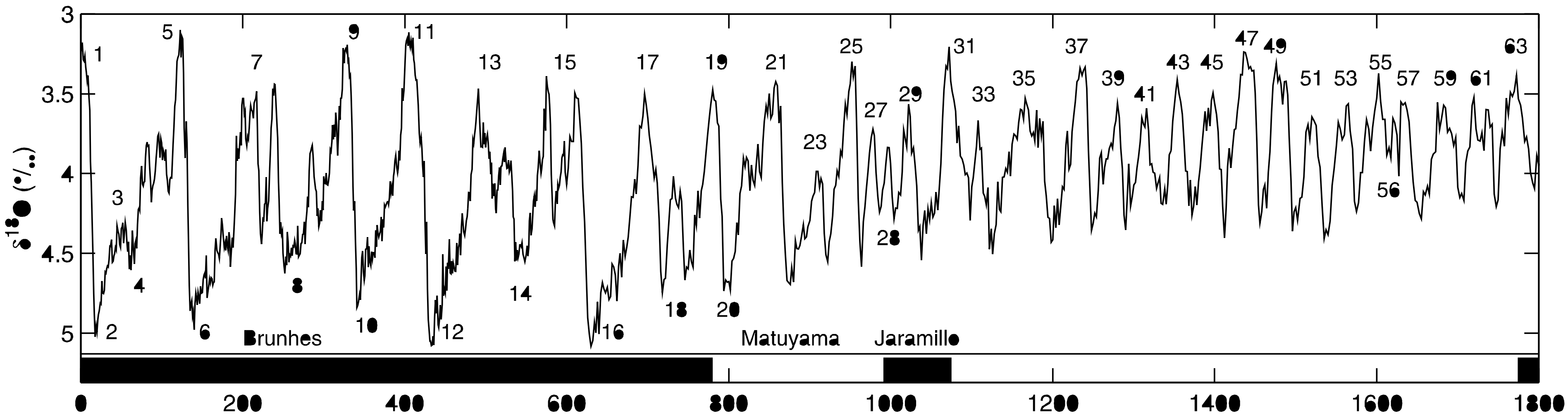
$$x \mapsto x^\alpha, \quad x \geq 0,$$

big jumps of the Lévy process govern asymptotic behavior

$\alpha < 1$: biggest jump responsible for exit

$\alpha > 1$: cumulative action of several large jumps responsible for exit

26. Paleoclimatic time series



Lisiecki, Raymo, Paleoceanography 2005 concentration variation of ^{18}O to ^{16}O taken from marine sediments at 57 globally distributed sites (e.g. Brunhes, Matuyama, Jaramillo):

global average temperature time series

basic feature:

- from 0 to -1 Myr periodicity $\sim$ 100 000 y
- from -1 Myr to -1.8 Myr periodicity $\sim$ 44 000 y
- Milankovich cycles: axial tilt (41 000 y) and eccentricity (100 000 y)