Does a topologically dense, bi-Lipschitz, Random Dynamical System on a compact metric space always have a unique invariant probability measure?

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The Problem

$(K, \mathcal{E}, \delta)$ compact, metric space, $\mathcal{M} = \{1, 2, ..., M\}$.
$f_i : K \to K, \quad i \in \mathcal{M} \quad$ continuous, $p_i > 0$, $\sum_{i \in \mathcal{M}} p_i = 1$.
Define $P : K \times \mathcal{E} \to [0, 1]$ by

$$P(x, E) = \sum_{i \in A(x,E)} p_i \quad where \quad A(x, E) = \{i : f_i(x) \in E\}$$

Since $f_i : K \to K$ continuous and K compact, there exists an invariant prob. measure μ associated to P.
Now consider the following two conditions:
1) **Condition R** : Each f_i is bijective and bi-Lipschitz.
2) **Condition M**:(Minimality). To every $\epsilon > 0$, there exists an integer N such that for any x and any y in K there exists a sequence $i_1, i_2, ..., i_N \in \mathcal{M}$ such that

$$\delta(f_{i_N}(f_{i_{N-1}}(...(f(_{i_1}(x))...)), y) < \epsilon.$$

Question: Does **Condition R** and **Condition M** $\Rightarrow \mu$ unique ?

Some helpful notations

Define $h : K \times \mathcal{M} \to K$ by $h(x,i) = f_i(x)$, $i \in \mathcal{M}$ define $q : \mathcal{M} \to (0,1]$ by $q(i) = p_i$, $i \in \mathcal{M}$, define $\mathcal{M}^n = \mathcal{M} \times \mathcal{M} \times .... \times \mathcal{M}$, n *times*, $n = 1, 2, ...$ set $i^n = (i_1, i_2, ..., i_n) \in \mathcal{M}^n$, define $h^n : K \times \mathcal{M}^n \to K$ recursively by $h^1(x,i) = h(x,i)$ and

$$h^{n+1}(x, i^{n+1}) = h(h^n(x, i^n), i_{n+1}),$$

and define $q^n : \mathcal{M}^n \to (0,1)$ by $q^n(i^n) = q(i_1)q(i_2)...q(i_n)$.
Condition B: There exist constants $0 < b < 1 < B$ such that

$$b\delta(x,y) < \delta(h^n(x,i^n), h^n(y,i^n)) \leq B\delta(x,y), \quad \forall i^n \in \mathcal{M}^n, \quad n = 1, 2, ...$$

Condition G: There exists a numbar $\alpha < 0$, a number $\beta > 0$ and an integer N such that if $0 < \delta(x,y) < \beta$ then

$$\sum_{i^N \in \mathcal{M}^N} \log\left(\frac{\delta(h^N(x,i^N), h^N(y,i^N))}{\delta(x,y)}\right) q^N(i^N) < \alpha.$$

Two theorems.

Theorem B. Cond. **R** + Cond. **M** + Cond.**B** $\Rightarrow \mu$ unique.
Theorem G. Cond. **R** + Cond. **M** + Cond. **G** $\Rightarrow \mu$ unique.
Q: Does Cond. **R** + Cond. **M** $\Rightarrow$ Cond **B** or Cond. **G**?
Theorem **B** easy. For Theorem **G** see [1] and [2].
Example: Let $K =$ the unit circle, let $\mathcal{M} = \{1, 2\}$, let f_1 = irrational rotation, let f_2 = "standard" circle map
$arg(f_2(exp(i\theta))) = \theta + a + bsin(\theta), \quad |b| < 1$
Then μ unique. (See [3].)

[1] T. Kaijser, Duke Math J, vol 45 ,no 2, pp 311 -349, 1978.
[2] T. Kaijser, Rev Roum Math Pure Appl , vol 26, no 8, pp 1075-1117, 1981.
[3] V.A.Kletsyn, M.B Nalskii, Func. Anal. Appl, vol 38, no 4, pp 267-282, 2004.