Annealed and quenched limit theorems for random expanding dynamical systems.

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January 19, 2015

Random dynamical systems.

Suppose

(i) (Ω, ν) is a probability space, and $\theta : \Omega \to \Omega$ is a measure preserving transformation

(ii) $(X, \mathcal{B})$ is a measurable space and to each $\omega \in \Omega$ we associate a map $T_\omega : X \to X$.

We define a skew-product transformation (RDS)

$$F : \Omega \times X \to \Omega \times X$$

$$F(\omega, x) = (\theta\omega, T_\omega x)$$

Iterating we consider orbits in X under a random composition of maps

$$T_{\theta^n\omega} \circ \ldots \circ T_\omega x$$

A probability measure μ on X is a stationary measure if and only if

$$\mu(A) = \int_\Omega \mu(T_\omega^{-1}(A))\, d\nu(\omega)$$

for each A.

I.I.D. case:

(Ω, ν) is a countable product space, $\Omega = S^{\mathbb{N}}$, ν is a product measure and θ is the full shift.

In this case $T_\omega = T_{\omega_1}$ for each $\omega \in \Omega$, with $\omega = (\omega_1, \omega_2, \ldots)$.

From now, we will always consider this i.i.d. situation and call this an i.i.d RDS.

Annealed and Quenched Limit Theorems.

Suppose $\phi : X \to \mathbb{R}$ is an observation. We may consider the time-series

$$\{\phi(x), \phi(T_{\omega_1}x), \phi(T_{\omega_2}T_{\omega_1}x), \ldots, \phi(T_{\omega_n}T_{\omega_{n-1}} \ldots T_{\omega_1}x), \ldots\}$$

as sampled with respect to the measure $\nu \times \mu$ (annealed dynamics) or by considering fixed realizations and sampling with respect to μ (quenched dynamics).

We may consider annealed and quenched probabilistic limit theorems such as the central limit theorem, large deviations, Erdós-Rényi limit laws and dynamical Borel-Cantelli lemmas.

Quenched dynamics is related to sequential dynamical systems but usually refers to sampling with respect to μ for ν a.e. realization of the random process.

Averaged transfer and Koopman operators

Assume X has a reference probability measure m and each T_ω is non-singular w.r.t. m (usually Lebesgue measure).
For a map $T : X \to X$ we define the Koopman operator of T by

$$U_T \phi = \phi \circ T$$

$U_T : L^\infty(m) \to L^\infty(m)$.

We define the transfer operator $P_T : L^1(m) \to L^1(m)$ by duality

$$\int \phi \psi \circ T \, dm = \int P_T(\phi) \psi \, dm$$

for all $\psi \in L^\infty(m)$, $\phi \in L^1(m)$.

For our RDS we introduce annealed (or averaged) transfer and Koopman operators:

For $\phi \in L^1(m)$, we define $P\phi$ by

$$P\phi(x) = \int_\Omega P_\omega \phi(x)\ d\nu(\omega)$$

For $\phi \in L^\infty(m)$, we define $U\phi$ by

$$U\phi(x) = \int_\Omega U_\omega \phi(x)\ d\nu(\omega)$$

where $P_\omega = P_{T_{\omega_1}}$, $U_\omega = U_{T_{\omega_1}}$ if $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots)$.

For all $n \geq 0$ and $\phi \in L^\infty(m)$, one has

$$U^n\phi(x) = \int_\Omega \phi(T_{\omega_n} \dots T_{\omega_1} x)\, d\nu(\omega)$$

because ν is a product measure.
Similarly

$$P^n\phi = \int_{\omega \in \Omega} p_\omega^n P_{T_{\omega_n} \dots T_{\omega_1}} \phi$$

where $p_\omega^n = p_{\omega_n} \dots p_{\omega_1}$ (could be replaced by an integral in case of infinitely many maps).

U is the dual operator of P, that is

$$\int_X P\phi(x)\psi(x)\ dm(x) = \int_X \phi(x)U\psi(x)\ dm(x)$$

for all $\phi \in L^1(m)$ and $\psi \in L^\infty(m)$.

An absolutely continuous probability measure μ is invariant or stationary if and only if its density $h = \frac{d\mu}{dm}$ is a fixed point of P.

Quasicompactness assumptions.

As in the case of a single map, in our applications we will require that the annealed operator P has good spectral properties on some Banach space of functions. More precisely, we show there exists a Banach space $(\mathcal{B}, \|.\|)$ such that :

1. $\mathcal{B}$ is compactly embedded in $L^1(m)$;
2. Constant functions lie in $\mathcal{B}$;
3. $\mathcal{B}$ is a Banach algebra : there exists $C \geq 0$ such that for all $f, g \in \mathcal{B}$ we have $fg \in \mathcal{B}$ with $\|fg\| \leq C\|f\|\|g\|$;
4. $\mathcal{B}$ is a complex Banach lattice : for all $f \in \mathcal{B}$, $|f|$ and $\bar{f}$ belong to $\mathcal{B}$;
5. $\mathcal{B}$ is stable under P : $P(\mathcal{B}) \subset \mathcal{B}$, and P acts continuously on $\mathcal{B}$;
6. P satisfies a Lasota-Yorke (or Doeblin-Fortet) inequality : there exist $N \geq 1$, $\rho < 1$ and $K \geq 0$ such that $\|P^N f\| \leq \rho\|f\| + K\|f\|_{L^1_m}$ for all $f \in \mathcal{B}$.

The Lasota-Yorke inequality (6) implies that P is quasi-compact on $\mathcal{B}$.

There exists $h \in \mathcal{B}$ with $Ph = h$, $h \geq 0$ and $\int_X h \, dm = 1$.

Let μ denote the stationary measure with density $h = \frac{d\mu}{dm}$.

If μ is mixing then h is unique.

Decay of correlations.

If P is quasi-compact on $\mathcal{B}$ and μ is mixing then for all $\phi \in \mathcal{B}$ and $\psi \in L^\infty(m)$:

$$(a) \left| \int_X \phi \, U^n \psi \, d\mu - \int_X \phi \; d\mu \int_X \psi \; d\mu \right| \leq C\lambda^n \|\phi\| \|\psi\|_{L^\infty_m}$$

If $\mathcal{B} \subset L^\infty(m)$

$$(b) \left| \int_X \phi \, U^n \psi \, d\mu - \int_X \phi \; d\mu \int_X \psi \; d\mu \right| \leq C\lambda^n \|\phi\| \|\psi\|_{L^1_m}$$

If $\mathcal{B} \subset L^\infty(m)$ and $h \geq a > 0$ then

$$(b) \left| \int_X \phi \, U^n \psi \, d\mu - \int_X \phi \; d\mu \int_X \psi \; d\mu \right| \leq C\lambda^n \|\phi\| \|\psi\|_{L^1_\mu}$$

Examples: Random Lasota-Yorke systems.

A Lasota-Yorke map is a piecewise C^2 map $T : [0,1] \to [0,1]$ for which $\lambda(T) := \inf |T'| > 0$.
We denote by P_T the transfer operator with respect to Lebesgue measure m.

$$P_T\phi(x) = \sum_{Ty=x} \frac{\phi(y)}{|T'(y)|}$$

for all $\phi \in L^1(m)$.
P_T acts on the space of functions of bounded variation, and we have the following Lasota-Yorke inequality :
For any $\phi \in \mathrm{BV}$, we have

$$\mathrm{Var}(P_T\phi) \leq \frac{2}{\lambda(T)}\mathrm{Var}(\phi) + A(T)\|\phi\|_{L^1_m}$$

where $A(T)$ is a finite constant depending only on T.

Let S be a finite set of Lasota-Yorke maps $T = \{T_i\}_{i \in S}$. Choose a map from S in an iid fashion, map T_i is chosen with probability p_i. This gives rise to a product measure ν on the set of compositions of maps $\Omega = S^{\mathbb{N}}$.
The system (Ω, ν, T) is called a random Lasota-Yorke system.
The annealed transfer operator associated to (Ω, T, ν) is given by

$$P = \sum_{i \in S} p_i P_{T_i}$$

The random Lasota-Yorke system (Ω, ν, T) is expanding in mean if

$$\Lambda := \sum_i \frac{p_i}{\lambda(T_i)} < 1$$

Examples:

(a) $T_1(x) = 2x,\ T_2(x) = \frac{x}{2}$ chosen with probabilities p, $1-p$ with $p > \frac{2}{3}$ has $\Lambda < 1$ and is expanding in mean.

(b) $T_1(x) = x + x^{1+\alpha}$, $0 < \alpha < 1,\ T_2(x) = \frac{3}{2}x + 1$ with $p = 1/2$, $1 - p = 1/2$ is expanding in mean.

Proposition (Pelikan, 1984)

If (Ω, ν, T) *is expanding in mean, then some iterate of the annealed transfer operator satisfies a Lasota-Yorke inequality on* BV*, the space of functions of bounded variation.*

Sketch of proof.
By subadditivity of the total variation,

$$\mathrm{Var}(P^n f) \leq 2\theta_n \mathrm{Var}(f) + A_n \|f\|_{L^1_m}$$

for all $n \geq 1$ and all $f \in \mathrm{BV}$ where

$$\theta_n = \sum_{\omega \in \Omega^n} \frac{p^n_\omega}{\lambda(T^n_\omega)}$$

$$A_n = \sum_{\omega \in \Omega^n} p^n_\omega A(T^n_\omega)$$

and $p^n_\omega = p_{\omega_n} p_{\omega_{n-1}} \dots p_{\omega_1}$.

For a single map,

$$\mathrm{Var}(P_T\phi) \le \frac{2}{\lambda(T)}\mathrm{Var}(\phi) + A(T)\|\phi\|_{L^1_m}$$

Since $\lambda(T^n_\omega) \ge \lambda(T_{\omega_1})\dots\lambda(T_{\omega_n})$, we have

$$\theta_n \le \sum_{\omega\in\Omega^n} \frac{p_{\omega_1}\dots p_{\omega_n}}{\lambda(T_{\omega_1})\dots\lambda(T_{\omega_n})} = \Lambda^n$$

Hence for n large enough, $2\theta_n < 1$.

The random Lasota-Yorke system (Ω, T, ν) is said to have the Random Covering (RC) property if for any non-trivial subinterval $I \subset [0,1]$, there exist $n \geq 1$ and $\omega \in \Omega^n$ such that $T_{\omega_n} \dots T_{\omega_1}(I) = [0,1]$.

Proposition

If (Ω, T, ν) is expanding in mean and has the random covering property, then (Ω, T, ν) is mixing, the density h of the unique absolutely continuous stationary measure μ is strictly bounded away from 0 and

$$\left| \int_X \phi \, U^n \psi \, d\mu - \int_X \phi \, d\mu \int_X \psi \, d\mu \right| \leq C \lambda^n \|\phi\| \|\psi\|_{L^1_\mu}$$

for all $\phi \in \mathcal{B}$ and $\psi \in L^1(\mu)$.

Prior results:

- S. Pelikan (TAMS, 1984)

-Showed that a finite collection of random Lasota-Yorke maps have an acip and gave results on number of ergodic components.

- V. Baladi (Communications in Mathematical Physics, 1997)

- studied annealed equilibrium states for iid random smooth uniformly expanding maps and described quenched equilibrium states

• J. Buzzi (Communication in Math Phys, 1999) assumed
(i) expansion on average for piecewise monotone and non-singular maps, $f_\omega : [0,1] \to [0,1]$ in sense that if $\delta(f_\omega) = \inf |f'_\omega|$ then

$$\lim_{K\to\infty} \int \log\min(\delta(f_\omega), K) d\nu > 0$$

(ii) random covering property
He proved exponential decay of random correlations i.e. decomposing μ as $\mu = \int_\Omega \mu_\omega d\nu$ we have for ν a.e. ω

$$|\int_X \phi \circ \psi \circ f_\omega^n \, d\mu_\omega - \int \phi d\mu_\omega \int_X \psi \, d\mu_\omega| \leq C(\omega)\rho^2 |\psi|_\infty \|\phi\|_{BV}$$

where $0 < \rho < 1$ and $C(\omega)$ is measurable.
-Buzzi has similar results for random affine maps $B_\omega : \mathbb{R}^n \to \mathbb{R}^n$ (mod $\mathbb{Z}$).
-Buzzi does not assume maps are chosen i.i.d, only ergodicity of $\theta : \Omega \to \Omega$.

• Ayyer, Stenlund and Liverani (2009)
Obtain annealed and quenched central limit theorems for smooth observations on iid linear hyperbolic maps of the torus.

• J. P. Conze, S. LeBorgne and M. Roger (2012)
Obtain quenched CLT for certain iid products of toral automorphisms.

• F. Tumel (UH thesis, 2012)
- proved the annealed central limit theorem for i.i.d uniformly expanding Rychlik maps (countable partition).

Annealed and quenched central limit theorems.

Let $\phi \in \mathcal{B}$ be a bounded real observation with $\int_X \phi \, d\mu = 0$.
Define X_k on $\Omega \times X$ by

$$X_k(\omega, x) = \phi(T_{\omega_k} \dots T_{\omega_1} x)$$

and

$$S_n = \sum_{k=0}^{n-1} X_k$$

Proposition

Under our quasicompactness assumptions:

(a) the limit $\sigma^2 = \lim_{n\to\infty} \frac{1}{n} E_{\nu\times\mu}(S_n^2)$ *exists, and is equal to* $\int_X \phi^2 \, d\mu + 2\sum_{n=1}^{+\infty} \int_X \phi U^n \phi \, d\mu$.

(b) $\sigma^2 = 0$ *if and only if there exists* $\psi \in \mathcal{B}$ *such that, for* ν*-a.e.* ω, $\phi = \psi - \psi \circ T_\omega$ μ*-a.e. (in case of finite number of maps,* ϕ *must be a coboundary for all maps).*

Denote by $\mathcal{M}_{\mathcal{B}}$ the set of all probability measures on X which are absolutely continuous w.r.t. m, and whose density lie in $\mathcal{B}$.

Theorem (Annealed CLT)

For every probability measure $\tilde{\mu} \in \mathcal{M}_{\mathcal{B}}$, the process $\frac{S_n}{\sqrt{n}}$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ under the probability $\nu \times \tilde{\mu}$.

Lemma (Berry-Esseen)

There exists $C > 0$ and $\rho < 1$ such that for all $t \in \mathbb{R}$ and $n \geq 0$ with $\frac{|t|}{\sqrt{n}}$ sufficiently small, and all $\nu \in \mathcal{M}_{\mathcal{B}}$, we have

$$\left|\mathbb{E}_{P \times \nu}(e^{i \frac{t}{\sqrt{n}} S_n}) - e^{-\frac{1}{2}\sigma^2 t^2}\right| \leq C \|\nu\| \left(e^{-\frac{\sigma^2 t^2}{2}} \left(\frac{|t| + |t|^3}{\sqrt{n}}\right) + \frac{|t|}{\sqrt{n}} \rho^n\right).$$

Proof is fairly standard. For $z \in \mathbb{C}$ define $P_z(f) = P(e^{z\phi} f)$, show

$$\int_\Omega \int_X e^{\frac{it}{\sqrt{n}} S_n(\omega,x)} f(x) dm d\nu = \int_X P_{\frac{it}{\sqrt{n}}}(f) dm$$

and use perturbation results.

Ayyer, Liverani and Stenlund (2009) used a technique from random walks in random scenery.
Let $\{T_\omega\}_{\omega\in\Omega}$ be a iid random dynamical system acting on X, with a stationary measure μ. Let $\phi : X \to \mathbb{R}$ be an observation with $\int_X \phi d\mu = 0$, and define $S_n = \sum_{k=0}^{n-1} \phi(T_\omega^k x)$.
We want to estimate

$$E_\nu(|\mu(e^{i\frac{t}{\sqrt{n}}S_n}) - e^{-\frac{t^2\sigma^2}{2}}|^2) = E_\nu(|\mu(e^{i\frac{t}{\sqrt{n}}S_n})|^2 - e^{-t^2\sigma^2} + C\frac{1+|t|^3}{\sqrt{n}}$$

We introduce a auxiliary random system defined as follows : the underlying probability space is (Ω, ν), but the auxiliary system acts on X^2, with associated maps $\hat{T}_\omega$ given by

$$\hat{T}_\omega(x, y) = (T_\omega x, T_\omega y)$$

Define a new observable $\hat{\phi} : X^2 \to \mathbb{R}$ by $\hat{\phi}(x, y) = \phi(x) - \phi(y)$, and denote its associated Birkhoff sums by $\hat{S}_n$.
Then $E_\nu(|\mu(e^{i\frac{t}{\sqrt{n}}S_n})|^2 = E_\nu[(\mu \times \mu)(e^{i\frac{t}{\sqrt{n}}\hat{S}_n})]$

Proposition

Assume there exists $\sigma^2 > 0$ and a constant $C > 0$ such that for all $t \in \mathbb{R}$ and $n \geq 1$ with $\frac{t}{\sqrt{n}}$ small enough :

(1) $\left| E_{\nu\times\mu}(e^{i\frac{t}{\sqrt{n}}S_n}) - e^{-\frac{t^2\sigma^2}{2}} \right| \leq C\frac{1+|t|^3}{\sqrt{n}}$*, and*

(2) $\left| E_{\nu\times(\mu\times\mu)}(e^{i\frac{t}{\sqrt{n}}\hat{S}_n}) - e^{-t^2\sigma^2} \right| \leq C\frac{1+|t|^3}{\sqrt{n}}$.

Suppose also that for $n \geq 1$ and $\epsilon > 0$:

(3) $(\nu \times \mu)\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) \leq Ce^{-C\epsilon^2 n}$.

Then, the quenched CLT holds : for ν-a.e. sequence $\omega \in \Omega$,

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\phi(T_{\omega^k}x) \to_\mu \mathcal{N}(0,\sigma^2).$$

To check this, conditions (1) and (3) are fairly immediate ((3) is a large deviations estimate).
With (2) the auxiliary system acts on a space whose dimension is twice the dimension of X, so we have to use more complicated functional spaces e.g. quasi-Hölder.
The other difficulty, less obvious at first glance, is that the asymptotic variance of $\hat{\phi}$ has to be twice the asymptotic variance of ϕ.
When all maps T_ω preserve the measure μ, this can be proved using the Green-Kubo formula.
But in the general situation, the stationary measure of the auxiliary measure can be different from $\mu \times \mu$, and it seems hard to compute the asymptotic variance of $\hat{\phi}$ from Green-Kubo formula.

Lemma

Using the same notations introduced above, assume that there exists $\sigma^2 > 0$ and $\hat{\sigma}^2 > 0$ such that

1. $\frac{S_n}{\sqrt{n}}$ *converges in law to* $\mathcal{N}(0, \sigma^2)$ *under the probability* $\nu \times \mu$,
2. $\frac{\hat{S}_n}{\sqrt{n}}$ *converges in law to* $\mathcal{N}(0, \hat{\sigma}^2)$ *under the probability* $\nu \times (\mu \times \mu)$,
3. *for a.e.* $\underline{\omega}$, $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \phi \circ T_{\underline{\omega}}^k$ *converges in law to* $\mathcal{N}(0, \sigma^2)$ *under the probability* μ.

Then $\hat{\sigma}^2 = 2\sigma^2$.

Large deviations

The spectral gaps enables us to show,

Theorem (Annealed Large Deviation Principle)

Suppose that $\sigma^2 > 0$. Then there exists a non-negative rate function c, continuous, strictly convex, vanishing only at 0, such that for every sufficiently small $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \times \nu(S_n > n\epsilon) = -c(\epsilon)$$

We have a quenched upper bound, with the same rate function for almost every realization :

Proposition (Quenched upper bound)

For every small enough $\epsilon > 0$ and for ν-almost every ω,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(\{x \in X : S_n(\omega, x) > n\epsilon\}) \leq -c(\epsilon)$$

Quenched CLT examples

Suppose that all maps T_ω are given by $T_\omega x = m_\omega x \mod 1$, where $m_\omega > 1$ is an integer. The transfer operator of this system satisfies (1)-(6) on the space BV, and is random-covering, so that assumption (1) and (3) hold automatically for any $\phi \in \mathrm{BV}$. On the other hand, the auxiliary two-dimensional system has a spectral gap on the quasi-Hölder space $V_1(X^2)$ and is also random covering. Since $\hat{\phi}$ belongs to $V_1(X^2)$, assumption (2) follows and the quenched CLT holds.

There exist piecewise non-linear expanding maps which preserves Lebesgue. Such a class of examples is provided by the Lorenz-like maps with both a neutral parabolic fixed point and a point where the derivative goes to infinity. The coexistence of these two behaviors allows the possibility for the map to preserve Lebesgue measure while being non-linear.

Haydn et al: gave an example of a class of intermittent type maps $T_\gamma : [-1,1] \to [-1,1]$, $\gamma > 1$, which preserve Lebesgue measure and are polynomially mixing of arbitrarily high order (depending on γ).

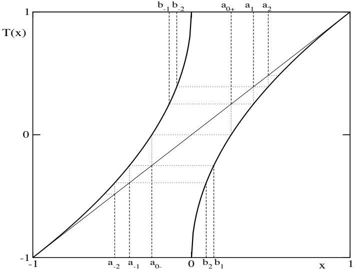
T_γ is implicitly defined by the equation

$$x = \begin{cases} \frac{1}{2\gamma}(1+T(x))^\gamma)^\gamma & \text{if } 0 \leq x \leq \frac{1}{2\gamma}; \\ T(x) + \frac{1}{2\gamma}(1-T(x))^\gamma & \text{if } \frac{1}{2\gamma} \leq x \leq 1. \end{cases}$$

and extended as an odd function so that $T(-x) = -T(x)$.

T preserves Lebesgue measure and has decay of correlations of rate $n^{-\frac{1}{\gamma-1}}$.

Figure: Graph of T



Suppose that $\Omega = \{0, 1\}$, F is the doubling map $Fx = 2x \mod 1$, and that T is such a parabolic map.

Then there exists $0 \leq p_\gamma < 1$ such that if F is iterated with probability p_γ with $p > p_\gamma$, then the quenched CLT holds for any Lipshitz observable ϕ.

Dynamical Borel-Cantelli lemmas

Suppose $T : X \to X$ is an ergodic transformation of (X, μ).

Question: Given a sequence of sets (A_n) (balls, rectangles,...) such that $\sum_j \mu(A_j) = \infty$, does $T^n(x) \in A_n$ infinitely often (i. o) for μ a. e. $x \in X$? If so, is there a quantitative rate?

If $p \in X$ and $A_n = B_{r_n}(p)$ is a sequence of nested balls about p of radius r_n, $r_n \to 0$, then the problem of whether $T^n(x) \in B_{r_n}(p)$ infinitely often for μ a. e. $x \in X$ is often called the shrinking target problem.

- Borel-Cantelli lemmas are used to establish almost sure limit laws.

We let

$$S_n(x) = \sum_{j=0}^{n-1} 1_{A_j}(T^j x)$$

$$E_n = \sum_{j=0}^{n-1} \mu(A_n)$$

We say a sequence of sets (A_n) satisfies the Strong Borel-Cantelli (SBC) property if $\sum_{j=0}^{\infty} \mu(A_j) = \infty$ and

$$\lim_{n \to \infty} \frac{S_n(x)}{E_n} = 1$$

for μ a. e. $x \in X$.

A sequence of sets (A_n) satisfies the Borel-Cantelli (BC) property if $S_n(x)$ is unbounded for μ a. e. $x \in X$.

Dynamical Borel-Cantelli Lemmas

Let (T, X, μ) be ergodic and suppose f_k is a sequence of non-negative measurable functions on (X, μ). Let $E_n = \sum_{j=1}^{n} \int f_j d\mu$.

$$|\int f_j \circ T^j f_i \circ T^i d\mu - \int f_j \int f_i| \leq Cp(j-i)\|f_i\||f_j|_{L^1}$$

where $\sum_n p(n) < \infty$ and $\|f_j\|$ is bounded. Then it is known that

$$\frac{1}{E_n} \sum_{j=1}^{n} f_j \circ T^j(x) \to 1$$

for μ a.e. $x \in X$.

Applying this result to the probability space $(\Omega \times X, \nu \times \mu)$, and using decay of correlations, we get :

Proposition (Annealed Borel-Cantelli)

Suppose (X, T, ν) is a random Lasota-Yorke system.
If ϕ_n is a sequence of non-negative functions in $\mathcal{B}$, with $\sup_n \|\phi_n\| < \infty$ and $E_n \to \infty$, where $E_n = \sum_{j=0}^{n-1} \int \phi_j \, d\mu$, then

$$\lim_{n\to\infty} \frac{1}{E_n} \sum_{j=0}^{n-1} \phi_j(S^j(x,\omega)) \to 1$$

for $\nu \times \mu$ a.e. $(\omega, x) \in \Omega \times X$.
The annealed version of the Strong Borel-Cantelli property implies a quenched version, namely

Proposition (Quenched Borel-Cantelli)

For ν-a.e. ω for μ-a.e. $x \in X$,

$$\lim_{n\to\infty} \frac{1}{E_n} \sum_{j=0}^{n-1} \phi_j(S^j(x,\omega)) \to 1$$

Shrinking target problem

In this case we consider a sequence of shrinking balls. Let $p \in X$ and let $B_n = B_n(p)$ be a sequence of nested balls about p.

We assume $\sum_{n=1}^{\infty} \mu(B_n)$ diverges.
Let $1_{B_n(p)}$ be the characteristic function of $B_n(p)$.
We look for a distributional limit for the sequence,

$$\frac{1}{a_n} \sum_{j=1}^{n} (1_{B_i} \circ T^i - \mu(B_i))$$

where a_n are suitable scaling constants.

Proposition (Annealed CLT)

Suppose (X, T, ν) is a random Lasota-Yorke system
Suppose $B_i := B_i(p)$ are nested balls of radius about a point $p \in X$, such that $\frac{C_1}{i^{\gamma_1}} \leq \mu(B_i(p)) \leq \frac{C_2}{i^{\gamma_2}}$ for some constants C_1, C_2 and $\gamma_2 < \gamma_1$.
Let $1_{B_n(p)}$ be the characteristic function of $B_n(p)$ and define

$$a_n^2 = E[\sum_{j=1}^{n}(1_{B_i} \circ T^i - \mu(B_i))^2]$$

Then $a_n \to \infty$ and

$$\frac{1}{a_n}\sum_{j=1}^{n}(1_{B_i} \circ F^i(\omega, x) - \mu(B_i)) \Longrightarrow_{\nu \times \mu} N(0, 1)$$

Proof: use a Martingale approach.

We map define the annealed transfer operator P and Koopman operator U by duality with respect to the stationary measure μ, instead of the measure m.
The stationary measure μ satisfies $\mu U = \mu$.

We define a Markov chain associated to μ and U :
Let $\Omega^\star = \{x = (x_0, x_1, x_2, \dots, x_n, \dots)\}$
There exists a unique probability measure μ_c on $\Omega^\star$ invariant under the one-sided shift τ,

$$\tau(x_0, x_1, \dots, x_n, \dots) = (x_1, x_2, \dots, x_n, \dots)$$

$\pi : (x_0, x_1, x_2, \dots) = x_0$ allows us to lift $\phi : X \to \mathbb{R}$ to $\phi_\pi : \Omega^\star \to \mathbb{R}$ by $\phi_\pi = \phi \circ \pi$.
We define the Koopman operator $\tilde{U}$ and the transfer operator $\tilde{P}$ associated to $(\Omega^\star, \tau, \mu_c)$.
We have $\tilde{P}^k \tilde{U}^k f = f$ and $\tilde{U}^k \tilde{P}^k f = \mathbb{E}_{\mu_c}(f|\mathcal{F}_k)$ for every μ_c integrable f, where $\mathcal{F}_k = \tau^{-k}\mathcal{F} = \sigma(x_k, x_{k+1}, \dots)$

If ϕ_n is a sequence of functions, $\int_X \phi_n d\mu = 0$, we may define for $n \geq 1$

$$w_n = P\phi_{n-1} + P^2\phi_{n-2} + \ldots P^n\phi_0$$

so that $w_1 = P\phi_0$, $w_2 = P\phi_1 + P^2\phi_0$, $w_3 = P\phi_2 + P^2\phi_1 + P^3\phi_0$ etc...

This converges as P has a spectral gap.

For $n \geq 1$ define

$$\psi_n = (\phi_n)_\pi - (w_{n+1})_\pi \circ \tau + (w_n)_\pi$$

where $\pi(x_0, x_1, \ldots, x_n, \ldots) = x_0$

$\tilde{P}\psi_n = 0$ and so $\mathbb{E}_{\mu_c}(\psi_n|\mathcal{F}_1) = 0$. Hence $\psi_n(\tau_n)$ is a reverse-martingale difference.

We now apply a theorem of Hall and Heyde.

Proposition (Theorem 3.2 [Hall and Heyde)

Let $\{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ *be a zero-mean square-integrable martingale difference array with differences* X_{ni} *and let* η^2 *be an almost sure finite random variable. Suppose that:*
(a) $\max_i |X_{ni}| \to 0$ *in probability;*
(b) $\sum_i X_{ni}^2 \to \eta^2$ *in probability;*
(c) $E(\max_i X_{ni}^2)$ *is bounded in n;*
(d) the σ*-fields are nested:* $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ *for* $1 \leq i \leq k_n$*,* $n > 1$*.*
Then $S_{ni} \to Z$ *(in distribution) where the random variable* Z *has the characteristic function* $E(\exp(-\frac{1}{2}\eta^2 t^2))$*.*

Open Questions.

- Does the quenched CLT hold for random Lasota Yorke systems expanding in mean (without a common invariant measure)?

- Can we obtain quenched large deviations and quenched almost sure invariance principle?

- Can we obtain similar results for non-iid compositions?