Attractors for nonautonomous random dynamical systems with an application to stochastic resonance

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- This work started with the study of the stardard model for the stochastic resonance: which is the RDS version of the phenomenon?

$$\mathrm{d}x = (\alpha x - \beta x^3)\mathrm{d}t + \mathcal{A}\cos \nu t \ \mathrm{d}t + \sigma \mathrm{d}W_t \quad \alpha, \beta, \sigma > 0, \quad x \in \mathbb{R}$$

  $(W_t)_{t\in\mathbb{R}}$ Gaussian noise

- We define a framework for random dynamical systems with a nonautonomous deterministic component (*nonautonomous RDS*) and in this setting we give definitions and results for dynamical entities such as global random attactors.
- We describe the stochastic resonance as a nonautonomous RDS.
- We show that the SR has a global random attractor and it is a random periodic orbit.
- We propose an indicator for the resonant regime which is naturally derived from the dynamical properties of RDS.

Definition

For random systems with a nonautonomous deterministic component, the dynamics depends on the initial time: in the definition of nonautonomous RDS, we keep the standard model for the evolution of noise and we add to the cocycle a variable accounting for the initial time.

Model of noise

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; a time set $\mathbb{T}$ ($\mathbb{R}, \mathbb{R}^+$ or $\mathbb{Z}$); a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$-measurable function $\theta : \mathbb{T} \times \Omega \to \Omega$ is called a ergodic dynamical system if the following four conditions are fulfilled.

(i) Initial value condition: $\theta_0 \omega = \omega$.

(ii) Group property: $\theta_{t+s}\omega = \theta_t(\theta_s \omega)$.

(iii) Invariance: $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$.

(iv) Ergodicity: $\theta_t A = A \Longrightarrow \mathbb{P}(A) \in \{0, 1\}$.

Let $X = \mathbb{R}^d$. Then

$\Phi : \mathbb{T} \times \mathbb{T} \times \Omega \times X \to X$, $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable, is a mapping with the cocycle properties:

(i) $\Phi(0, \tau, \omega, x) = x$ for all $\tau \in \mathbb{T}$, $\omega \in \Omega$ and $x \in X$,

(ii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s \omega, \Phi(s, \tau, \omega, x))$ for all $t, s, \tau \in \mathbb{T}$, $\omega \in \Omega$ and $x \in X$.

We will indifferently write $\Phi(t + s, \tau, \omega, x)$ and $\Phi(t + s, \tau, \omega)x$.

Periodic random dynamical system

A special case of nonautonomous RDS is a periodic RDS. We say that the RDS is periodic if there exists a number $T > 0$ such that

$$\Phi(t, \tau + T, \omega, x) = \Phi(t, \tau, \omega, x) \text{ for all } t, \tau \in \mathbb{T},\ \omega \in \Omega,\ x \in X$$

Remarks:

- Another possibility to describe nonautonomous RDS is by removing the invariance hypothesis for $\mathbb{P}$ under θ in the model for noise evolution.
- Previous work on nonautonomous RDS has been done by T. Caraballo, P.Kloeden, B. Wang...

Example: discrete-time case

Consider a metric space X and four homeomorphisms $h^i_j : X \to X$, $i, j = 0, 1$. We want to study the random dynamics if h^i_j is used with probability p_j.

Generation of a discrete-time RDS

(i) The ergodic dynamical system θ is given by:
$\Omega := \{\omega = (\dots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \dots) : \omega_i \in \{0,1\}\}$.
$\mathbb{P}(I_{x_1,\dots,x_n}) := \prod_{i=1}^n p_{x_i}$ (defined on cylinder sets with $x_i \in \{0,1\}$).
$\mathcal{F} = \sigma(\text{cylinder sets})$.
θ: left shift.

(ii) $\varphi(1, m, \omega)x := (h^{m \mod 2}_{\omega_0})(x)$ for $m \in \mathbb{N}$.

The cocycle property is

$$\varphi(n-1, m+1, \theta_1\omega)\varphi(1, m, \omega)x =$$
$$= (h^{m+n-1 \mod 2}_{\omega_{n-1}} \circ \dots \circ h^{m+1 \mod 2}_{\omega_1}) \circ ({h_{\omega_0}}^{m \mod 2}(x)) = \varphi(n, m, \omega)x \,.$$

Example: continuous-time case

Given a 1-dim SDE

$$\mathrm{d}x = f(x,t)\mathrm{d}t + \sigma \mathrm{d}W_t, \sigma > 0, \quad x \in \mathbb{R}$$

$(W_t)_{t \in \mathbb{R}}$ is a Wiener process.

- The model for the noise is:
  - $\Omega := C_0(\mathbb{R},\mathbb{R}) := \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$ equipped with the compact-open topology and the Borel σ-algebra $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$
  - $\mathbb{P}$ is the Wiener probability measure on $(\Omega, \mathcal{F})$.
  - The evolution of noise is described by the Wiener shift $\theta : \mathbb{R} \times \Omega \to \Omega$, defined by $\theta(t, \omega(\cdot)) := \omega(\cdot + t) - \omega(t)$.
- Let $X(t, \tau, \omega, x)$ be the stochastic flow of the SDE, for initial time τ and initial condition $x \in \mathbb{R}$; the cocycle is defined by

$$\Phi(t, \tau, \theta_\tau \omega, x) =_{def.} X(t + \tau, \tau, \omega, x)$$

Generalising RDS concepts to the nonautonomous case

Nonautonomous random sets

We call *nonautonomous random set* a set-valued function $M : \mathbb{T} \times \Omega \mapsto 2^X, \ (\tau, \omega) \mapsto M(\tau, \omega)$, taking values in the subsets of a metric space (X, d), with $(\tau, \omega) \mapsto d(x, M(\tau, \omega))$ measurable for each $x \in X$.[1]

The set $M(\tau, \omega)$ is the (τ, ω)-fiber of M.
We say that M is *compact* if $M(\tau, \omega)$ is compact.

Nonautonomous invariant random sets

A random set M is *invariant* with respect to the RDS iff.

$$\Phi(t, \tau, \omega) M(\tau, \omega) = M(\tau + t, \theta_t \omega)$$

for all $t, \tau, \in \mathbb{T}$ and for almost all $\omega \in \Omega$.

[1] $\text{dist}(C, D) := \sup_{c \in C} d(c, D)$ is the *Hausdorff semi-distance* of C and D.

Periodic orbit

- An invariant random set $p : \mathbb{R} \times \Omega \mapsto X$ whose fibers are points is called random periodic orbit if there exists a positive time T such that

$$\Phi(T, \tau, \omega) p(\tau, \omega) = p(\tau, \theta_T \omega)$$

for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$

Nonautonomous random attractors

We define attracting sets in terms of pullback.

Nonautonomous attracting sets

A random set A is attracting for a random set M iff.

$$\lim_{t\to\infty} \operatorname{dist}\left(\Phi(t,\tau-t,\theta_{-t}\omega)M(\tau-t,\theta_{-t}\omega), A(\tau,\omega)\right) = 0$$

for all $\tau\in\mathbb{T}$, for almost all $\omega\in\Omega$

Nonautonomous random global attractors

Let $\mathcal{H}$ be a family of random sets: a random set $A\in\mathcal{H}$ is a global $\mathcal{H}$-attractor if

- A is invariant;
- A attracts in the pullback sense every $M\in\mathcal{H}$

Remark For the purpose of this talk, we are interested in global random attractors for the family of all deterministic bounded sets, i.e.

$$\lim_{t\to\infty} \operatorname{dist}\left(\Phi(t,\tau-t,\theta_{-t}\omega)B, A(\tau,\omega)\right) = 0 \text{ for all } \tau\in\mathbb{T}, \text{ for almost all } \omega\in\Omega$$

for all bounded $B\subset X$.

Existence of random global attractors

A classical result relating compact absorbing sets and global attractors still holds in the nonautonomous case [2].

Nonautonomous absorbing set

A random set B is absorbing for a random set M if, for almost all $\omega \in \Omega$, there exists a time $T(M, \tau, \omega) > 0$ such that for all $\tau \in \mathbb{T}$, for all $t \geq T(M, \tau, \omega)$

$$\Phi(t, \tau - t, \theta_{-t}\omega)M(\tau - t, \theta_{-t}\omega) \subset B(\tau, \omega)$$

[2] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probability Theory and Related Fields **100** (1994), no. 3; F. Flandoli and B. Schmalfuß, *Random attractors for the 3-D stochastic Navier-Stokes equation with mulitiplicative white noise*, Stochastics and Stochastics Reports **59** (1996).

Ω-limit sets

Definition (Ω-limit sets)

Given a random set $M(\tau,\omega)$, we define the Ω-limit

$$\Omega_M(\tau,\omega) := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \Phi(t,\tau - t,\theta_{-t}\omega)M(\tau - t,\theta_{-t}\omega)}$$

or equivalently:

$$\Omega_M(\tau,\omega) := \{y \in X : \exists\ t_n \to \infty, \exists\ x_n \in M(\tau - t_n,\theta_{-t_n}\omega) \text{ s.t. } y = \\ = \lim_{n\to\infty} \Phi(t_n,\tau - t_n,\theta_{-t_n}\omega)x_n\}$$

Existence of global random attractors

Proposition

Let $\mathcal{H}$ be a family of random sets, inclusion-closed [3] and $K \in \mathcal{H}$ a compact random set absorbing every $M \in \mathcal{H}$ (including itself). Then Ω_K is a global $\mathcal{H}$-attractor and it is unique.

Remark In particular, if K is a compact random set absorbing all deterministic bounded sets, then there exists a unique global random attractor for all deterministic bounded sets and it is compact. (For this version of the proposition, it is not necessary for K to be a deterministic bounded set.).

[3] We say that a family $\mathcal{H}$ of random sets is *inclusion-closed* if for each $M \in \mathcal{H}$ $M(\tau, \omega) \neq \emptyset$ for all (τ, ω), and if $M_2 \in \mathcal{H}$ and $\emptyset \neq M_1(\tau, \omega) \subset M_2(\tau, \omega)$, $\forall (\tau, \omega)$, imply that $M_1 \in \mathcal{H}$.

Application: stochastic resonance.

The concept of stochastic resonance broadly indicates a class of phenomena in nonlinear systems where a weak 'signal' can be be amplified and optimized by the presence of noise.

It was originally introduced to explain the (almost) periodic recursion of glaciations (R.Benzi and G. Parisi, C. Nicolis...): the periodic effect of the changes in time of the eccentricity of the earth's orbit around the sun is amplified by environmental noise given by atmosferic fluctuations.

Other phenomena in physics and biology can be explained by the role played by the noise in the amplification of a signal (ring laser, crayfish, barn owl...)

- Recent mathematical work on the subject by:

  N. Berglund, B. Gentz, S. Herrmann, P. Imkeller, I. Pavlyukevich, D. Peithmann.

Description of the phenomenon

- A simple model for stochastic resonance is given by a damped particle in a periodically oscillating double-well potential in the presence of noise: when applying a periodic forcing, the double-well potential is tilted asymmetrically up and down, periodically raising and lowering the potential barrier.
- For weak periodic forcing, the noise strenght can be tuned so that noise-induced hopping between the wells become synchronised with the periodic forcing.
- In this case the average waiting time between two noise-induced interwell transitions is comparable with half the period of the forcing. Outside the resonant range of parameters, for increasing noise strength, the periodicity is lost and the hopping becomes increasingly random.

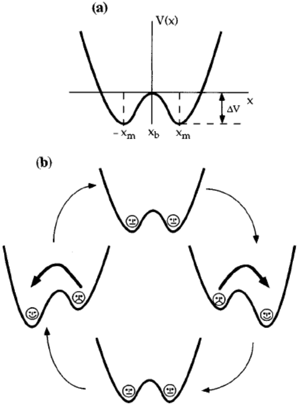


Figure 1 : Effect of the periodic forcing on a double-well potential.
L. Gammaitoni et al , *Stochastic resonance*, Reviews of Modern Physics **70** (1998)

SDE for the stochastic resonance

The phenomenon appears also in the one-dimensional approximation of this model, which describes the dynamics of an overdamped particle:

$$\mathrm{d}x = \left(\alpha x - \beta x^3\right)\mathrm{d}t + \mathcal{A}\cos\nu t\;\mathrm{d}t + \sigma\mathrm{d}W_t \quad \alpha, \beta, \sigma > 0, \quad x \in \mathbb{R}$$

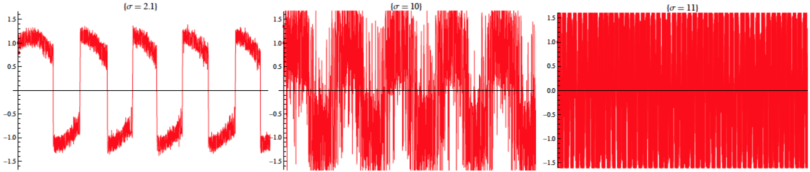


Figure 2 : Paths for the SDE at increasing value of noise

Existence of a global attractor

The SDE

$$\mathrm{d}x = (\alpha x - \beta x^3)\mathrm{d}t + \mathcal{A}\cos\nu t\ \mathrm{d}t + \sigma \mathrm{d}W_t \quad \alpha, \beta, \sigma > 0, \quad x \in \mathbb{R}$$

generates a nonautonomous RDS (θ, Φ), where the cocycle Φ is given by the stochastic flow, as defined earlier.

We prove that the RDS has a compact nonautonomous random set absorbing all deterministic bounded sets. This implies the existence of a nonautonomous random global attractor.

The main idea in the proof is to change variable in order to transform the SDE into a RODE and get asymptotic estimates for its solution.

Change of variable and RODE

Given the Ornstein-Uhlenbeck SDE

$$\mathrm{d}y = -y\mathrm{d}t + \sigma \mathrm{d}W_t \qquad y \in \mathbb{R} \tag{1}$$

The pathwise solution for the equation is given by

$$Y(t, \tau, \omega, y_\tau) = y_\tau e^{-t} + \sigma e^{-t} \int_\tau^t e^r dW_r$$

The pathwise pullback limit gives

$$O_t(\omega) = \sigma e^{-t} \int_{-\infty}^t e^r dW_r$$

unique stationary solution, which can be seen as a random variable defined on Ω.

Change of variable and RODE

- Let $Z_t = X_t - O_t$, where X_t is shorthand for a pathwise solution of the SDE for the stochastic resonance
- Z_t satisfies the random ordinary differential equation

$$\frac{dz}{dt} = f(z + O_t(\omega), t) + O_t(\omega)$$

$f(x,t) = \alpha x - \beta x^3 + \mathcal{A} \cos \nu t$. We associate to the RODE a nonautonomous RDS; Ω is the Wiener space, the cocycle is given by

$$\Psi(s, \tau, \omega) z =_{def.} Z(\tau + s, \tau, \theta_{-\tau}\omega, z)$$

for all $t, \tau, \in \mathbb{R}, \omega \in \Omega, z \in \mathbb{R}$

Dissipativity and integrability conditions

The following conditions hold for f:

i *Dissipativity condition.* There exist constants $L_1, L_2 \geq 0$ such that $(x_1 - x_2)\big(f(x_1,t) - f(x_2,t)\big) \leq L_1 - L_2 \left|x_1 - x_2\right|^2$.

ii *Integrability condition.* $\exists C_0 > 0$ s.t. $\int_{-\infty}^{t} \exp^{c\,r} \left|f(u(r),r)\right|^2 dr < \infty$, for all $0 < c < C_0$ and u continous in $\mathbb{R}$ with sub-exponential growth.

Given $Z(t, \tau, \omega, z)$ solution of (21) we have, because of the dissipativity condition (omitting the dependence on τ, ω, and initial conditions):

$$\frac{dZ_t^2}{dt} = 2Z_t\big(f(X_t,t) + O_t\big) \leq 2\big(L_1 - L_2 Z_t^2\big) + L_3 Z_t^2 + \frac{1}{L_3}\big(f(O_t,t) + O_t\big)^2$$

for any $L_3 > 0$

$$\Downarrow$$

$$\frac{dZ_t^2}{dt} \leq -C_1 Z_t^2 + C_2 + C_3\big(f(O_t,t) + O_t\big)^2$$

where $C_1 = L_2 - L_3 > 0$ can be taken smaller than C_0 in the integrability condition. For brevity, let's $C_2 + C_3\big(f(x,t) + x\big) := F(x,t)$

Estimates

The differential inequality leads to

$$|\Psi(s,\tau,\omega)z|^2 \leq |z|^2 e^{-C_1 s} + e^{-C_1(s+\tau)} \int_{\tau}^{s+\tau} e^{-C_1 r} F(O_r(\theta_{-\tau}\omega), r) dr$$

- Given an initial condition $x \in C$, bounded set, for the stochastic resonance SDE, the corresponding initial condition z is such that $z + O_\tau(\omega) \in C$, which defines a random set $\widetilde{C}$;
- $|z|^2 e^{-C_1 s} \leq 1$ for s 'big enough'.

Then for times s big enough in the pullback we have:

$$|\Psi(s,\tau-s,\theta_{-s}\omega)z|^2 \leq 1 + e^{-C_1\tau}\int_{\tau-s}^{\tau} e^{-C_1\ r} F(O_r(\theta_{-s}\circ\theta_{s-\tau}(\omega)),r)dr$$

In the limit $s\to\infty$

$$\lim_{s\to\infty}|\Psi(s,\tau-s,\theta_{-s}\omega)z|^2 \leq 1 + e^{-C_1\tau}\int_{-\infty}^{\tau} e^{-C_1\ r} F(O_r(\theta_{-\tau}\omega),r)dr := R(\tau,\omega)^2$$

where the integral is well defined because of the integrability condition.

Existence of a compact absorbing set

There exists $\widetilde{T}(C,\tau,\omega)$ s.t. for $s > \widetilde{T}(C,\tau,\omega)$:

$$\Psi(s,\tau-s,\theta_{-s}\omega)C(\tau,\omega) \subset B\left(R(\tau,\omega)\right)$$

ball with radius $R(\tau,\omega)$

$$\Downarrow$$

for any bounded set C there exists $T(C,\tau,\omega)$ s.t. for $s > T(C,\tau,\omega)$

$$\Phi(s,\tau-s,\theta_{-s}\omega)C \subset B\left(O_\tau(\omega),R(\tau,\omega)\right)$$

$B\left(O_\tau(\omega),R(\tau,\omega)\right)$ ball of radius $R(\tau,\omega)$ centered in $O_\tau(\omega)$.

- $B\left(O_\tau(\omega),R(\tau,\omega)\right)$ is a random compact set absorbing all deterministic bounded sets for the RDS for the stochastic resonance.
- This implies that the RDS as a random global attractor for the family of deterministic bounded sets.
- The attractor is a periodic, compact and connected random set, each fiber is an interval in $\mathbb{R}$.

In order to prove that the global random attractor is a periodic orbit, we need to characterise measures for the nonautonomous RDS

Invariant and periodic invariant measures for the nonautonomous RDS.

The skew product for the nonautonomous RDS is the map
$\Theta : \mathbb{T} \times \mathbb{T} \times \Omega \times X \mapsto \mathbb{T} \times X \times \Omega$

$$(t, \tau, \omega, x) \mapsto (\tau + t, \theta_t \omega, \Phi(t, \tau, \omega)x)$$

We say that $\mu : \mathbb{T} \times \mathcal{F} \otimes \mathcal{B} \mapsto [0, 1]$ is an invariant measure for the nonautonomous RDS if

- for all t, $\mu(t, \cdot)$ is a measure on $\Omega \times X$ such that for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$, for all t, τ
$$\mu(\Theta(t, \tau, A)) = \mu(\tau, A)$$
- $\pi_\Omega \mu(t, \cdot) = \mathbb{P}$ where $\pi_\Omega \mu(t, \cdot)$ denotes the marginal on $(\Omega, \mathcal{F})$

We say that an invariant measure μ is periodic invariant for a nonautonomous periodic RDS if $\exists T > 0$ such that for all t

$$\mu(t, \cdot) = \mu(t + T, \cdot)$$

Disintegration of measures

We write μ_τ for $\mu(\tau, \cdot)$.

μ_τ can be uniquely 'disintegrated' into a family $\mu_{(\tau,\omega)}$ of probability measures on X as follows

$$\mu_\tau(A) = \int_\Omega \mu_{(\tau,\omega)}(A_\omega) d\mathbb{P}(\omega)$$

where $A_\omega = \{x \in X : (x, \omega) \in A\}$.

μ is invariant iff.

$$\Phi(t, \tau, \omega)\mu_{(\tau,\omega)} = \mu_{(\tau+t,\theta_t\omega)}$$

for all $t, \tau, \in \mathbb{T}$ and for almost all $\omega \in \Omega$

An invariant μ is periodic invariant iff.

$$\Phi(T, \tau, \omega)\mu_{(\tau,\omega)} = \mu_{(\tau,\theta_T\omega)}$$

for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$

Periodic stationary measures for the Markov semigroup

Let ρ_τ, $\tau \in \mathbb{R}$, be a measure on X, invariant for the non-homogeneous Markov semigroup associated to the SDE for the stochastic resonance, i.e. if $P(t, \tau, x, B)$ is the probability transition for the process

$$\rho_{\tau+t}(B) = \int_X P(\tau, x, t, B) d\rho_\tau(x)$$

for all $B \in \mathcal{B}(X)$.

We say that the family $\{\rho_\tau\}$ is a periodic stationary measure (or periodic invariant for the semigroup) if

$$\rho_{\tau+T} = \rho_\tau \quad \text{for all } \tau$$

The attractor is a point

To prove that the global attractor is a random point, we generalise to the nonautonomous case a result in:

- H. Crauel and F. Flandoli, *Additive noise destroys a pitchfork bifurcation*, Journal of Dynamics and Differential Equations **10** (1998), no. 2, 259–274.

Proposition

i There exists a compact global attractor A for the RDS, which is a periodic random set with period T.

ii The RDS is order-preserving i.e. if $x \geq y$ then $\Phi(t,\tau,\omega,x) \geq \Phi(t,\tau,\omega,y)$, for all t,τ, for almost all ω

If there exist a unique T-periodic stationary measure $\{\rho_t\}$, $t \in \mathbb{R}$, $A(\tau,\omega)$ is a point for all (τ,ω).

The proposition is based on the fact that the existence of a T-periodic stationary measure implies the existence of a T-periodic invariant measure for the RDS.

This can be proved by applying to the discrete RDS defined by the time T map $\tilde{\Phi}(n, \omega, x) := \Phi(nT, 0, \omega, x)$ results in:

- H. Crauel, *Markov measures for random dynamical systems*, Stochastics and Stochastic Reports **37** (1991), no. 3, 153–173
- H. Crauel *Extremal exponents of random dynamical systems do not vanish*, Journal of Dynamics and Differential Equations **2** (1990), no. 3, 245–291.

Then:

- there exists a unique T-periodic family of measures $\{\rho_\tau\}$ invariant for the non-homogeneous Markov semigroup associated to the stochastic resonance;
- the pullback limit

$$\mu_{(\tau,\omega)} := \lim_{t\to\infty} \Phi(t, \tau - t, \theta_{-t}\omega)\rho_{\tau-t}$$

exists for all $\tau \in \mathbb{T}$, for almost all $\omega \in \Omega$ and defines a T-periodic measure for the RDS.

An indicator for the resonance.

A natural indicator for the resonance properties of the system can be derived by density of the distribution of the periodic attractor.

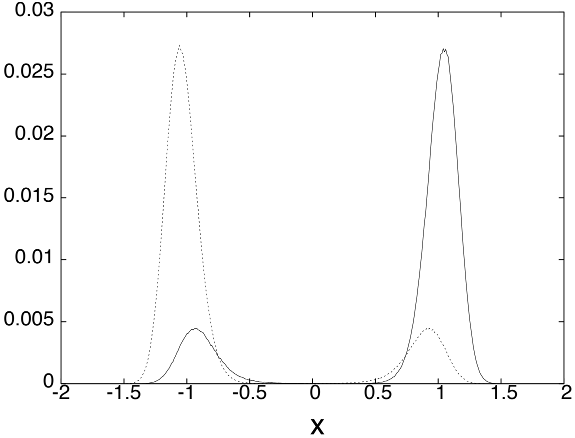
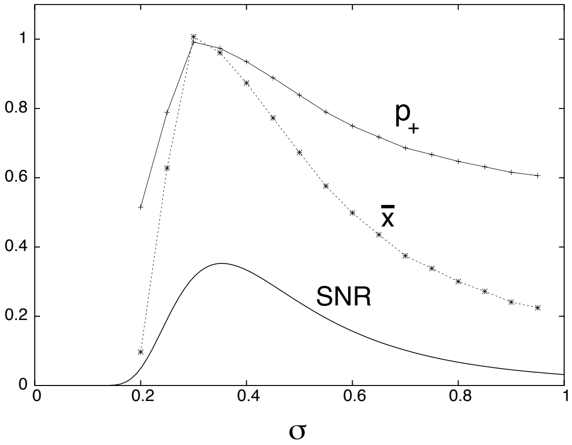


Figure 3 : The figure shows the density of the distribution of the periodic attractor in a resonant regime at a time τ and $\tau + \frac{T}{2}$ (dotted and continuous lines).

- The strong asymmetry in the distribution is progressively less marked at increasing noise, until eventually the system exits the resonant regime.
- The absolute value of the difference between the integrals for the density function in the two intervals defined by the potential wells is maximised for set of parameters in the resonant regime.
- This value can be used as an indicator (p_+ in the following slide) for the stochastic resonance, naturally deduced by the dynamical properties of the RDS (and providing an extimate of how many particles move, in time, between wells).

Comparison with other indicators

Preliminary tests show differences between the response of p_+, the signal-to-noise ratio and the mean value of x.[4]



[4] See *L. Gammaitoni et al., Reviews of Modern Physics (1998)* for the definition of the indicators.

Conclusions

- The nonautonomous RDS generated by the equation

$$dx = (\alpha x - \beta x^3)dt + \mathcal{A}\cos\nu t\ dt + \sigma dW_t \quad \alpha, \beta, \sigma > 0, \quad x \in \mathbb{R}$$

  has a random global attractor for all deterministic bounded sets.

- The attractor is a periodic orbit and the disintegration of the periodic invariant measure for the RDS is a Dirac measure.

- We derive an indicator for the resonant regime from the distribution of the global attracting point: the efficency of this indicator in comparison with other indicators for the stochastic resonance will be further investigated.

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Thanks for your attention!