# Asymptotics for random intermittent maps of the interval

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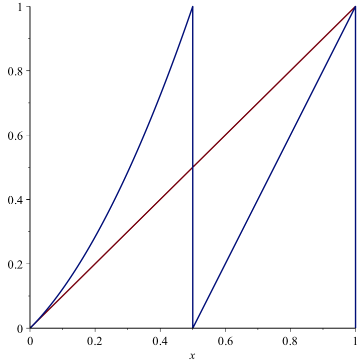
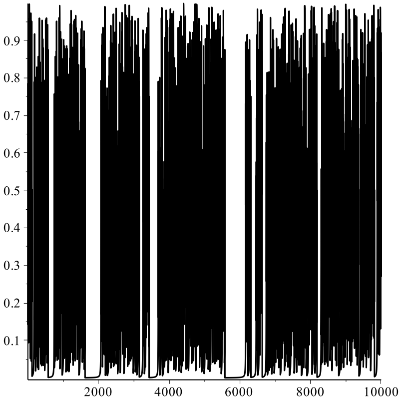
An example of an *intermittent* map.
Fix $0 < \alpha < \infty$. Set

$$T_\alpha(x) := \begin{cases} x + 2^\alpha x^{1+\alpha} & x \in [0, 1/2) \\ 2x - 1 & x \in [1/2, 1) \end{cases}$$

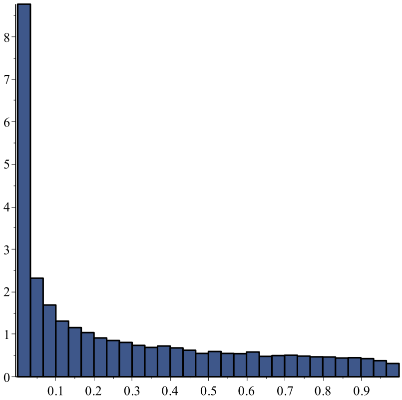


Orbits mostly spread 'uniformly' throughout $[0,1)$ interspersed with short periods getting 'stuck' near the neutral fixed point at $x=0$.




The periods of getting stuck are the *intermittencies*.

A histogram derived from the orbit gives a picture of an *invariant density* for the map $T_\alpha$. It is known that the density has an order $O(x^{-\alpha})$ singularity near $x = 0$.



# History 1

- Pianigiani (Israel J. Math. 1980) studied maps with finitely many indifferent fixed points; classified points as regular (admits a finite ACIM) or irregular (admits only a $\sigma-$finite ACIM). In our example

$$T_\alpha \text{ regular} \iff 0 < \alpha < 1$$

- Thaler (also IJM 1980) A focus on number theoretic examples.

# History 2

- Liverani, Saussol, Vaienti (ETDS 1999) established regularity properties of the invariant density for $T_\alpha$ and proved sub-exponential decay of correlation in the case of regular fixed point (i.e. $0 < \alpha < 1$ ) and finite ACIM:

$$Cor_n(g, f) := \int (g \circ T^n) f \, d\mu - \int g \, d\mu \int f \, d\mu$$

$$|Cor_n(g, f)| \leq C(f)||g||_\infty (\log n)^{\frac{1}{\alpha}} n^{1-\frac{1}{\alpha}}$$

for $f \in \mathcal{C}^1$ and $g \in L^\infty$. $\mu$ is the ACIM

- The maps $T_\alpha$ above are known as LSV-maps. Related: Pomeau-Manneville maps.

# History 3

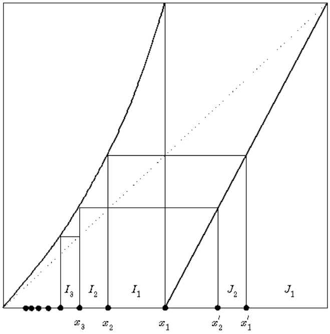
- LS Young (Israel J. Math 1999) introduced Markov Extensions (now called Young Towers) to study maps like $T_\alpha$ by inducing (very different approach from LSV). For LSV maps she showed

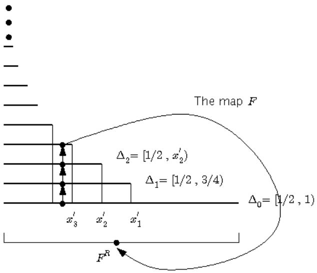
$$|Cor_n(g,f)| \le C(f)||g||_\infty n^{1-\frac{1}{\alpha}} \tag{1}$$

  for $f \in \mathcal{C}_\beta$ and $g \in L^\infty$.

This has become (one of) the industry standard methods for analysis of nonuniformly hyperbolic maps.

# Young Towers




If $R(x) = n + 1$ then $F^R(x) = T_\alpha^{n+1}(x) \in [1/2, 1]$

$T_\alpha$ is a factor of $F : \Delta \to \Delta$
($F$ is also called a Markov extension of $T_\alpha$).

- $\Delta = \cup \Delta_n$ and for $(x, n) \in \Delta_n$ define $\pi(x, n) = T_\alpha^n(x) \in [0, 1]$
- We have

$$\begin{array}{ccccc} & \Delta & \xrightarrow{F} & \Delta & \\ \pi & \downarrow & & \downarrow & \pi \\ & [0,1] & \overrightarrow{T_\alpha} & [0,1] & \end{array}$$

- Note, $\pi$ is (typically) countable to one.
- If $\nu \circ F^{-1} = \nu$ then $\mu = \nu \circ \pi^{-1}$ is invariant for $T_\alpha$ on $[0, 1]$.

- The existence of finite ACIM $\nu \sim m$ for $F$ and the rate of correlation decay are related to the summability and rate of tail sum decay (resp.) for

$$\sum_k m(\Delta_k)$$

- Therefore, important quantities to estimate are $x_n$ and $x'_n - 1/2$ controlling the (Lebesgue $= m$) measure of $\Delta_k$.
- There exist $c, C$ such that for all $n > 0$

$$cn^{-\frac{1}{\alpha}} \leq x_n \leq Cn^{-\frac{1}{\alpha}}.$$

Since $x'_n - 1/2 = \frac{1}{2}x_n$ we have the same asymptotics for the $x'_n - 1/2$
- Also need distortion control on the return map $F^R$ from $\Delta_0$ to $\Delta_0$: $\exists\theta < 1$ and constant $C$ such that

$$\left|\frac{DF^R(x)}{DF^R(y)} - 1\right| \leq C\theta^{d(F^R(x),F^R(y))}$$

Summary of LS Young tower result.

Assume $\sum m(\Delta_n) = \int_{\Delta_0} R(x)dm(x) < \infty$ and the distortion condition holds. Then,

- there is an $F-$ invariant ACIM $\nu$ on $\Delta$ equivalent to $m$.

Extend $R$ to $\hat{R}$ on the tower $\Delta$ as hitting time to $\Delta_0$.

- If $m\{\hat{R} > n\} = O(n^{-\gamma})$ then $Cor_n(f, g) = O(n^{-\gamma})$ for $\theta-$ Hölder data.
- Central limit theorem holds (for regular data) if $\gamma > 1$

Note that in the tower notation,

$$m\{\hat{R} > n\} = \sum_{k>n} \sum_{j>k} J_j$$

Specifically

- For $T_\alpha$, we obtain the required distortion estimates and easily compute

$$\sum_{j>k} m(J_j) = (x_k' - \frac{1}{2}) = \frac{1}{2}x_k \sim k^{-\frac{1}{\alpha}}$$

so

$$m\{\hat{R} > n\} \sim \sum_{k>n} k^{-\frac{1}{\alpha}} \sim n^{1-\frac{1}{\alpha}}$$

Correlation exponent is $\gamma = \frac{1}{\alpha} - 1 > 0$ within the parameter range $0 < \alpha < 1$

# History 4

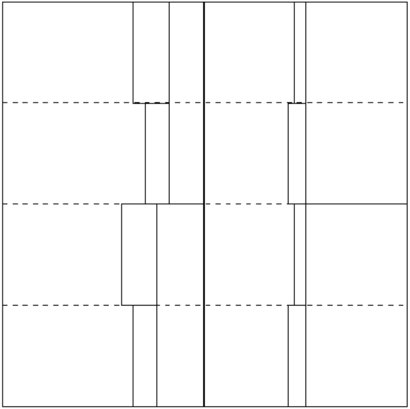
For $T = T_\alpha$ and $Cor_n(f,g) = O(n^{1-\frac{1}{\alpha}})$

- H. Hu, 0. Sarig and S. Gouëzel (2002-2004) showed the rate is sharp when $0 < \alpha < 1$.
- Central limit theorems hold when $\nu = \frac{1}{\alpha} - 1 > 1 \Leftrightarrow 0 < \alpha < \frac{1}{2}$
- When $\alpha \geq 1$ the ACIM is $\sigma-$ finite. Melbourne &Terhesiu (Invent. 2012) established mixing and correlation decay estimates for suitably normalized correlation.

Now we consider *random intermittent maps*

- Let $0 < \alpha < \beta < \infty$ and $T_\alpha, T_\beta$ two intermittent LSV maps and consider
$$T := (T_\alpha, T_\beta, p_1, p_2)$$
the associated random dynamical system.
- We can represent T as a deterministic skew product on $[0,1] \times [0,1)$ by
$$S(x,y) = (T_{\alpha(y)}(x), \sigma(y))$$
Here
$$\sigma(y) = \begin{cases} \frac{y}{p_1} \text{ if } y \in [0, p_1) \\ \frac{y-p_1}{p_2} \text{ if } y \in [p_1, 1) \end{cases} ; \alpha(y) = \begin{cases} \alpha \text{ if } y \in [0, p_1) \\ \beta \text{ if } y \in [p_1, 1) \end{cases}$$
This is just the independent $(p_1, p_2)-$ shift

In order to apply Young's construction, we need analogues of the intervals $I_n$ and $J_n$ from the single map case. Since the position $x_n = x_n(y)$ (similarly $x_n'(y)$), instead of intervals we see the following picture:



$\Delta_0 = [1/2, 1) \times [0, 1)$ and the return sets $I_n$ and $J_n$ are unions of rectangles stacked 'vertically'

The key estimates are again

$$\begin{aligned}
\sum_{k>n}\sum_{j\geq k} m\times m(J_j) &= \sum_{k>n}\sum_{j\geq k} E_y(x_j'(y) - x_{j+1}'(y)) \\
&= \frac{1}{2}\sum_{k>n}\sum_{j\geq k} E_y(x_j(\sigma y) - x_{j+1}(\sigma y)) \\
&= \sum_{k>n}\sum_{j\geq k} E_y(x_j(y)) - E_y(x_{j+1}(y)) \\
&= \sum_{k>n} E_y(x_k(y))
\end{aligned}$$

So we need to estimate the expected position of $x_k(y)$ over the randomizing variable $y$.

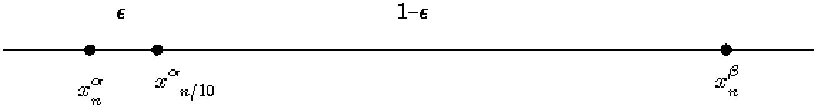
## Proposition

*For* $0 < \alpha \leq \beta < \infty$, $E_y(x_n(y)) = O(n^{-\frac{1}{\alpha}})$

(Sketch) Suppose $x_n^\alpha$ and $x_n^\beta$ denote the distinguished points constructed from each of the single maps. It is pretty much obvious (and easy to prove) that for all $n, y$

$$x_n^\alpha \leq x_n(y) \leq x_n^\beta$$

Now consider the following picture.




Note that $x_n^\alpha = O(n^{-\frac{1}{\alpha}}) = x^\alpha_{\frac{n}{10}} << x_n^\beta$ for large $n$.

In particular, if there are at least $n/10$ occurrences of $\alpha$ in

$$\alpha(y), \alpha(\sigma y), \dots \alpha(\sigma^{n-1} y)$$

then we have

$$x_n^\alpha \leq x_n(y) \leq x^\alpha_{\frac{n}{10}}$$

But large deviations estimates show this holds for a typical point y;

**Theorem (Hoeffding, 1963)**

*Let $X_k$ be an independent sequence of RV, with*

$$0 \leq X_k \leq 1 \;\; \forall k$$

*Set* $\bar{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$ *and* $E_n = E(\bar{X}_n)$ *Then for each* $0 < t < 1 - p_1$

$$\mathbb{P}\{\bar{X}_n - E_n > t\} \leq \exp(-2nt^2)$$

Specifically, Hoeffding's estimate implies, for any fixed $0 < p_0 < p_1,\ t := p_1 - p_0$, the probability that the string

$$\alpha(y), \alpha(\sigma y), \ldots \alpha(\sigma^{n-1} y)$$

has fewer than $p_0 n$ occurrences of $\alpha$ is at most

$$\exp[-2n(p_1 - p_0)^2]$$

Even a course upper bound like $x_n(y) \leq 1/2$ for $y$ in this set of exceptional points allows us to estimate

$$E_y(x_n(y)) = O(n^{-\frac{1}{\alpha}})$$

### Theorem

*Let* $0 < \alpha < \beta \leq 1$. *Then there is an ACIPM* $d\mu = hdm$ *for the random system T and we have the following decay of correlation result:*

$$|Cor_n(g, f)| \leq C(f)||g||_\infty n^{1-\frac{1}{\alpha}}$$

*provided f is Hölder on* $[0, 1)$ *and* $g \in L^\infty$

Note we cannot ignore the distortion requirement on the map $F^R$ in application of Young's theorem! This is where the additional restriction $\beta \leq 1$ comes from.

This result can be compared to Gouëzel (ETDS 2007) where a "curve" of randomizers is prescribed:

- $\alpha = \alpha(y)$ is $C^2$ on $[0, 1)$ and $0 < \alpha(y) < 1$.
- $\alpha_{min} = \min_y \alpha(y)$ is obtained at a unique point $y_{min}$
- $\alpha_{max} = \max_y \alpha(y) \leq 3/2 \cdot \alpha_{min}$
- $\alpha''(y_{min}) > 0$
- The skew product is $S(x, y) := (T_{\alpha(y)}, 4y (\text{mod } 1))$
- In this case it is proved that

$$E_y(x_n(y)) = O((\sqrt{\log n}/n)^{\frac{1}{\alpha_{min}}})$$

# Gouëzel RDS

- Note the appearance of the $\sqrt{\log n}$ factor in Gouëzel's result. This is sharp, since he actually proves almost everywhere convergence

$$\left(\frac{n}{\sqrt{\log n}}\right)^{\frac{1}{\alpha_{min}}} x_n(y) \to cst$$

- On the other hand it is not clear (as noted by Gouëzel) that the constraint $\alpha_{max} \leq 3/2 \cdot \alpha_{min}$ is essential. We do not need this for our example, thanks to Hoeffding. If we try to repeat the proof of Gouëzel using a large deviations estimate, we get the same upper bound on $\alpha_{max}$.
- Gouëzel also obtains CLT and stable limit laws for $1/2 \leq \alpha < 1$ under extra technical assumptions

# A few remarks

- Our estimates $E_y(x_n) \sim n^{-\frac{1}{\alpha}}$ remain true for all $0 < \alpha < \beta < \infty$ but we are unable extend the required distortion estimates past $\beta = 1$. This is probably a technical obstruction (since there is no such issue with a single map).
- In view of the above, if we replace $T_\alpha$ with a piecewise affine version (e.g. Gaspard-Wang, PNAS 1988) then we can obtain finite ACIM and correlation decay for $0 < \alpha < 1$ and $\alpha < \beta < \infty$.
- What about the infinite measure case, when $1 \leq \alpha < \beta < \infty$?