Block Triangularization of Matrix Cocycles

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Outline

Cocycles

Definition

Let $(X, \mathcal{B}, \mu, T)$ be a probability space with a measure-preserving map T (the 'base dynamics'). A (measurable) **matrix cocycle** is a measurable map $A : \mathbb{Z} \times X \to M_n(\mathbb{F})$ satisfying the following:

1. $A(0, x) = I$, for all $x \in X$.
2. $A(n + m, x) = A(m, T^n(x))A(n, x)$, for all $n \in \mathbb{N}$ and $x \in X$. ('Cocycle' property)

If T is invertible and $A(1, x) \in GL_n(\mathbb{F})$ is invertible for all x, then we may require the cocycle property to hold for all $n, m \in \mathbb{Z}$.

Note, as usual, that A is generated by its time-one map $A(1, x)$.

MET

Theorem (Multiplicative Ergodic Theorem, Invertible version)

Let $(X, \mathcal{B}, \mu, T)$ be a probability space equipped with an invertible, ergodic, measure-preserving map T. Let $A : X \to GL_n(\mathbb{R})$ be a measurable map generating a cocycle, such that

$$\int_X \log^+ \|A(x)\| \, d\mu < \infty, \ \int_X \log^+ \left\|A(x)^{-1}\right\| \, d\mu < \infty.$$

Then there exist $\lambda_1 > \lambda_2 > \cdots > \lambda_k > -\infty$, positive integers $m_1, m_2, \ldots, m_k$, and measurable families of subspaces $V_1(x), V_2(x), \ldots, V_k(x)$ of $\mathbb{R}^n$ such that for almost every $x \in X$:

1. $\bigoplus_{i=1}^k V_i(x) = \mathbb{R}^n$, $V_i(x) \cap V_j(x) = \{0\}$, *and* $\dim(V_i) = m_i$;
2. *For* $v \in V_j \setminus \{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|A(n,x)v\| = \lambda_j$ *and* $\lim_{n\to\infty} \frac{1}{n} \log \|A(-n,x)v\| = \lambda_j$;
3. $A(x)V_i(x) = V_j(T(x))$.

Measurable Selection

Walters, 1993: Measurable subspaces $x \mapsto V_i(x) \iff$ measurable basis vectors $x \mapsto v_{i,j}(x),\ j = 1 \ldots m_i$.

Collecting these vector-valued functions together in a matrix $C(x)$ allows us to 'conjugate' the cocycle by taking $C(T(x))^{-1}A(1,x)C(x)$. Hence...

MET Redux

Theorem (Equivalent Formulation of the MET, Invertible Case)

Let $(X, \mathcal{B}, \mu, T)$ and A be as before. Then there exist $\lambda_1 > \lambda_2 > \cdots > \lambda_k > -\infty$, positive integers $m_1, m_2, \ldots, m_k$, and a measurable function $C : X \to GL_n(\mathbb{R})$ such that for almost every $x \in X$:

1. $C(T(x))^{-1}A(x,1)C(x)$ *is block diagonal, with the i^{th} block of size m_i;*
2. *For $v \neq 0$ in the columnspace of the i^{th} block,* $\lim_{n\to\infty} \frac{1}{n} \log \|A(n,x)v\| = \lambda_j$ *and and* $\lim_{n\to\infty} \frac{1}{n} \log \|A(-n,x)v\| = \lambda_j$.

Remark

The equivalence condition is encompassed by the block diagonalization. C can be chosen to be orthogonal.

Motivation

Oseledets, 1968: Extended the base space for the cocycle by $SO_n(\mathbb{R})$ and constructed a triangular cocycle for this larger space, in order to use nice properties of such a cocycle. Perhaps it is possible to triangularize without extending the base?

As well, in analogy with single matrices, an upper triangular form seems to be a refinement of a block triangularization. So purely from an aesthetic perspective, one might hope to accomplish something like this.

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Remark

Upper-triangularization implies the existence of an equivariant family of 1-D subspaces.

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Remark

These results are only about real-valued conjugation and normal forms. They do not precisely refine the MET.

A Better Question

With single matrices, we like to triangularize over $\mathbb{C}$, as it is always possible, unlike real triangularization.

Question

Can we always block upper-triangularize a matrix cocycle, over $\mathbb{C}$? That is, find $C : X \to GL_n(\mathbb{C})$ such that $C(T(x))^{-1}A(1,x)C(x)$ is block upper-triangular over $\mathbb{C}$?

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Remark

If one uses the MET before attempting this, the problem reduces to triangularizing each block separately. (We have not attempted to describe anything like a complex Lyapunov exponent.)

An Answer

Answer

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Answer

Not always!

An Almost-Example

Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\mathcal{B} =$ Borel sets, $\mu = \lambda$ (normalized Lebesgue measure), and $T : \mathbb{T} \to \mathbb{T}$, $T(x) = x + \eta$, $\eta \in \mathbb{Q}^c$. Define

$$A(1, x) = \begin{bmatrix} \cos(\pi x) & -\sin(\pi x) \\ \sin(\pi x) & \cos(\pi x) \end{bmatrix}.$$

Then the cocycle A cannot be upper-triangularized over $\mathbb{R}$, but may be triangularized over $\mathbb{C}$.

Remark

A has Lyapunov exponents equal to 0, hence its Oseledets splitting is trivial.

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For the first part, apply a theorem by Thieullen, 1997. Alternatively, we may proceed bare-handed, so to speak.

The ‘Invariant Ponytail’ argument

A can be thought of as acting on $Gr_1(\mathbb{R}^2)$, the Grassmannian of 1-D subspaces of $\mathbb{R}^2$, which is homeomorphic to $[0, \pi)$ (or $\mathbb{T}$). There, it acts as

$$R(x, y) = (x + \eta, y + x),$$

which Furstenburg (and others) have proved to be ergodic with respect to Lebesgue measure.

For contradiction, assume that A may be triangularized; this means there is an equivariant family of subspaces $x \mapsto V(x)$, which implies, since

$$A(1, x)V(x) = V(T(x)),$$

that the graph of V on $\mathbb{T}$ is invariant under R:

$$R(x, V(x)) = (x + \eta, V(x) + x) = (x + \eta, V(x + \eta)).$$

The 'Invariant Ponytail' Argument

Finally, computing

$$R(x, V(x) + h) = R(x + \eta, V(x) + h + x) = (x + \eta, V(x + \eta) + h)$$

shows that any vertical translate of the graph is invariant, and hence that there exists an invariant set of positive measure. This is a contradiction, which shows that an equivariant family of real 1-D subspaces cannot exist.

Complex Case

However,

$$C(x) \equiv \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

diagonalizes every single matrix in the range of A, so there is no extra work to obtain $C(x)$ such that

$$C(T(x))^{-1}A(x)C(x) = \begin{bmatrix} e^{\pi x} & 0 \\ 0 & e^{-\pi x} \end{bmatrix}.$$

An Actual Example

Let $\alpha \in [0,1)$ be irrational, and consider the same base dynamics as before. Let

$$A(1,x) = \begin{cases} \begin{bmatrix} \cos(\pi\alpha) & -\sin(\pi\alpha) \\ \sin(\pi\alpha) & \cos(\pi\alpha) \end{bmatrix} & x \in [0, 1-\eta), \\ \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & x \in [1-\eta, 1). \end{cases}$$

This matrix cocycle also has 0 for both Lyapunov exponents, but this time there is no obvious way to triangularize it over $\mathbb{C}$. In fact, is it even triangularizable over $\mathbb{R}$?

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Answer

No, to both!

Outline of Proof

1. Instead of $Gr_1(\mathbb{R}^2)$, we are dealing with $Gr_1(\mathbb{C}^2)$, which is homeomorphic to $\bar{\mathbb{C}}$ (or the Riemann Sphere $\mathbb{S}^2$). Furthermore, A acts as either a rotation (about the polar axis), or as an inversion about the unit circle (that is, a flip over the equator).
2. We see that A leaves pairs of circles invariant: those circles equidistant from the equator (including the two poles).
3. Assuming there is an equivariant family of subspaces, we see that either these subspaces lie on a pair of circles, or on the equator.

Outline of Proof

4. In the case of two circles, project down to a two-point extension and prove that the resulting map is ergodic, yielding the same contradiction as before.
5. In the case of one circle, project down to an interval extension and proceed as earlier. This is also how one shows that the cocycle is not upper-triangularizable over $\mathbb{R}$.

Remark

The resulting dynamics is much more difficult to show to be ergodic. We utilized a result by Schmidt, 1976 (Theorem 12.8), which took much work to prove.

A... Better? Example

Let X be the full shift on $\{0, 1\}$, with cylinder set σ-algebra, product measure μ, and left shift σ. Define a cocycle:

$$A(1, x) = \begin{cases} \begin{bmatrix} \cos(\pi\alpha) & -\sin(\pi\alpha) \\ \sin(\pi\alpha) & \cos(\pi\alpha) \end{bmatrix} & x_0 = 0, \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & x_0 = 1. \end{cases}$$

We may show that the cocycle generated by A *also* may not be triangularized, by the same overall scheme as before, but without needing to utilize a powerful theory.

A Conjecture

Conjecture

The set matrix cocycles into $O(2)$ which cannot be triangularized over $\mathbb{C}$ is generic, with respect to a reasonable topology.

Approach: Break the cocycle into its rotation part and its flipping part, and work on the factors.

The End

Thank you!

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