Entropy for control problems and random escape rates

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Outline

Section 1

Motivation

Digitally networked control systems

Examples:

- Automated traffic control, e.g. vehicle platoons
- Sensor networks, e.g. smart cities
- Unmanned aerial vehicles
- Telerobotics
- ...

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Problem in digitally networked systems

Information cannot be transmitted instantaneously, lossless and with arbitrary precision. This raises the question about the smallest rate of information above which a given control task can be solved.

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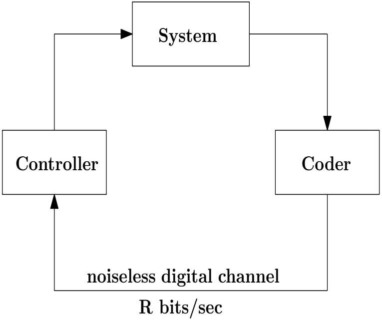
Simple control problem

Invariance of a subset of the state space (example: vehicle platoons)

The simplest setting



Explanation

System Deterministic, discrete or continuous time

Coder Encodes the state by a symbol from a finite alphabet at discrete times $k\tau$, $k = 0, 1, 2, \ldots$

Controller Generates open-loop controls on a finite time interval $[0, \tau]$

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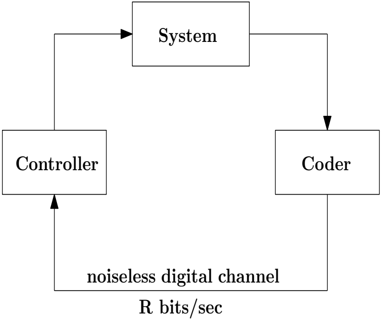

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Control objective

Invariance of a compact subset of the state space

Section 2

Controlled invariance and invariance entropy

Control-affine systems

A continuous-time control system on a smooth manifold M of the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad u \in \mathcal{U} = L^\infty(\mathbb{R}, U),$$

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with a compact and convex control range $U \subset \mathbb{R}^m$ is called control-affine. Assuming that all solutions are defined on the whole time axis, such a system generates a skew-product flow (control flow)

$$\phi : \mathbb{R} \times (\mathcal{U} \times M) \to \mathcal{U} \times M, \quad \phi_t(u,x) = (\theta_t u, \varphi(t,x,u)).$$

Here $\varphi(t,x,u)$ is the solution of the ODE for the control function u and initial value x and

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad (\theta_t u)(s) = u(t+s),$$

denotes the shift flow. With the weak*-topology, $\mathcal{U}$ is compact and metrizable and ϕ is continuous.

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denotes the shift flow. With the weak*-topology, $\mathcal{U}$ is compact and metrizable and ϕ is continuous.
Hypothesis: $Q \subset M$ is a compact and controlled invariant set, i.e.,

$$\forall x \in Q \; \exists u \in \mathcal{U} \text{ with } \varphi(\mathbb{R}_+, x, u) \subset Q.$$

Definition of invariance entropy

Definition (Colonius, K., 2009)

For $\tau > 0$ a set $\mathcal{S} \subset \mathcal{U}$ is called (τ, Q)-spanning if

$$\forall x \in Q \ \exists u \in \mathcal{S} \text{ with } \varphi([0,\tau], x, u) \subset Q.$$

Let $r_{\mathrm{inv}}(\tau, Q)$ be the minimal cardinality of such a set. Then the invariance entropy of Q is defined as

$$h_{\mathrm{inv}}(Q) := \lim_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{inv}}(\tau, Q).$$

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Intuition

If the controller receives n bits per time unit, it can generate at most 2^n different control functions. Hence, the number of control functions necessary to solve the control task on a finite time interval is a measure for the needed information.

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Equivalent: Feedback entropy (Nair, Evans, Mareels, Moran 2004)

Section 3

The entropy formula for uniformly hyperbolic control sets

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A set $E \subset M$ is called a chain control set if E is the projection of a maximal ϕ-invariant chain transitive set on $\mathcal{U} \times M$.

Fact: If the system is sufficiently regular, every control set with nonempty interior is contained in a chain control set.

Uniform hyperbolicity (without center bundle)

Assume that M is endowed with a Riemannian metric.

Definition

Let $Q \subset M$ be a compact set, controlled invariant in forward and in backward time. Q is called uniformly hyperbolic (without center bundle) if there is a continuous invariant splitting

$$T_x M = E^-_{u,x} \oplus E^+_{u,x}$$

for all (u, x) with $\varphi(\mathbb{R}, x, u) \subset Q$ such that with $\varphi_{t,u} = \varphi(t, \cdot, u)$,

$$\begin{aligned} |(\mathrm{d}\varphi_{t,u})_x v| &\leq c^{-1} \mathrm{e}^{-\lambda t} |v|, \quad v \in E^-_{u,x}, \\ |(\mathrm{d}\varphi_{t,u})_x v| &\geq c \mathrm{e}^{\lambda t} |v|, \quad v \in E^+_{u,x}, \end{aligned}$$

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Note: This only makes sense for explicitly time-dependent systems!

Entropy formula

Theorem (Da Silva, K., 2014)

Consider a control-affine system with a chain control set Q such that

- $f_0, f_1, \ldots, f_m$ are $\mathcal{C}^\infty$,
- Q is compact and uniformly hyperbolic,
- the Lie algebra rank condition holds on int $Q \neq \emptyset$ and
- for each $u \in \mathcal{U}$ there is a unique $x(u) \in Q$ with $\varphi(\mathbb{R}, x(u), u) \subset Q$.

Then Q is the closure of a control set and

$$h_{\mathrm{inv}}(Q) = \inf_{(u,x)} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \left| \det \left(\mathrm{d}\varphi_{\tau,u}|_{E^+_{u,x}} : E^+_{u,x} \to E^+_{\phi_\tau(u,x)} \right) \right|,$$

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Note: Here we use a slightly modified definition of invariance entropy.

Qualitative version

The proof of the entropy formula shows:

Qualitative result

There exists no strategy to make a uniformly hyperbolic chain control set invariant that cannot be beaten by the "simple" strategy of stabilizing the system at a periodic orbit inside the chain control set. In particular, it suffices to take the infimum over the ϕ-periodic points (u, x) in the entropy formula.

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Analogy

The topological entropy of a dynamical system on a uniformly hyperbolic set is completely determined by the periodic trajectories in this set (R. Bowen, 1970).

Tools used in the proof

For the upper estimate we combine control-theoretic methods for nonlinear systems with classical tools from the hyperbolic theory of dynamical systems (e.g., shadowing).

For the lower estimate we adapt techniques used in the estimation of escape rates for classical dynamical systems, developed by R. Bowen, D. Ruelle and L.–S. Young.

In particular, the proof doesn't use any ergodic theory!

Section 4

Relation to random escape rates

Lower estimate in terms of a uniform escape rate

Let m be a smooth measure on M. For each $\tau > 0$ and $u \in \mathcal{U}$ put

$$Q(u,\tau) := \{x \in Q \ : \ \varphi([0,\tau],x,u) \subset Q\}.$$

If $\mathcal{S} \subset \mathcal{U}$ is (τ, Q)-spanning, then $Q = \bigcup_{u \in \mathcal{S}} Q(u,\tau)$, implying

$$m(Q) \leq \sum_{u \in \mathcal{S}} m(Q(u,\tau)) \leq \#\mathcal{S} \cdot \sup_{u \in \mathcal{U}} m(Q(u,\tau)).$$

This leads to the estimate

$$h_{\mathrm{inv}}(Q) \geq -\liminf_{\tau \to \infty} \frac{1}{\tau} \sup_{u \in \mathcal{U}} \log m(Q(u,\tau)) =: -\mathrm{ER}_{\mathrm{unif}}(Q).$$

Estimate via random escape rates

Let μ be a shift-invariant probability measure on $\mathcal{U}$. Then the control flow on $\mathcal{U} \times M$ becomes a random dynamical system and the quantity

$$\mathrm{ER}_{\mu}(Q) := \lim_{\tau \to \infty} \frac{1}{\tau} \int_{\mathcal{U}} \log m(Q(u,\tau)) \mathrm{d}\mu(u)$$

is called a random escape rate, see

(1) P.–D. Liu. *Random perturbations of Axiom A basic sets.* J. Stat. Physics 90, 1–2 (1998), 467–490

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Conjecture (variational principle for escape rates)

$$\mathrm{ER}_{\mathrm{unif}}(Q) = \sup_\mu \mathrm{ER}_\mu(Q),$$

the supremum taken over all (ergodic) shift-invariant probabilities μ.

Uniform hyperbolicity (with center bundle)

Consider again a control-affine system.

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for all $t \geq 0$ with constants $c \in (0, 1]$ and $\lambda > 0$, and $\dim E^0_{u,x} \equiv 1$.

Note: This generalizes uniform hyperbolicity for flows.

Random escape rates in the uniformly hyperbolic case

According to a result of P.–D. Liu we can expect that

$$\begin{aligned} \mathrm{ER}_\mu(Q) &= P_{\mathrm{top}}(\alpha^+) \\ \alpha^+(u,x) &:= -\log\left|\det\left(\mathrm{d}\varphi_{1,u}|_{E^+_{u,x}} : E^+_{u,x} \to E^+_{\phi_1(u,x)}\right)\right| \end{aligned}$$

and that there exists a unique ϕ-invariant measure (equilibrium state) ν with projection μ s.t. (using the MET)

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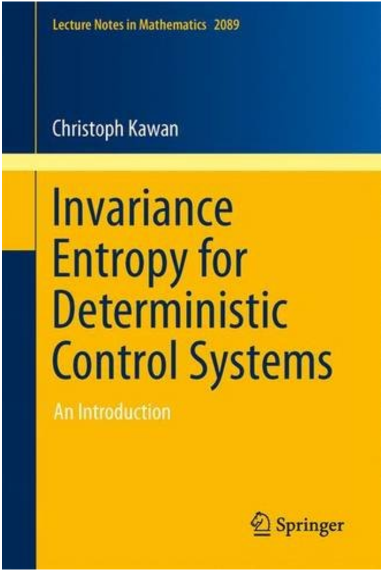
Question:

What about the case with center bundle?

Nair, Evans, Mareels, Moran: *Topological feedback entropy and nonlinear stabilization*. IEEE Trans. Aut. Control 49 (2004), 1585–1597

Colonius, K: *Invariance entropy for control systems*. SIAM J. Control Optim. 48 (2009), no. 3, 1701–1721

Da Silva, K: *Invariance entropy of hyperbolic control sets*. Revised version submitted to DCDS A. Preprint on arXiv (2014)



Thank you for your attention!