Almost sure invariance principle for sequential and non-stationary dynamical systems (see paper @ arxiv)

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Summary

Goal

ASIP (Almost Sure Invariance Principle):

- a.s. approximation of a sequence of RV's

$$S_n = X_1 + X_2 + \cdots + X_n$$

  by Brownian motion (details later)
- more refined statistical property
- implies CLT, LIL, WIP, etc.

Main idea:

- $S_n = f_1 \circ T_1 + f_2 \circ (T_2 T_1) + \cdots + f_n \circ (T_n \ldots T_1)$, subtract its mean
- "decay of correlations" $\implies$ approximation by a reverse martingale [Gordin (1969), ..., Conze & Raugi (2007)]
- Cuny & Merlevéde (2014): reverse martingale (with some properties) $\implies$ ASIP

What is new:

- previous results: **vector**-valued ASIP for Birkhoff sums over (non-uniformly) hyperbolic systems [Berger (1990), Melbourne & Nicol (2009), Gouëzel (2010)]

$$S_n = f \circ T + \cdots + f \circ T^n$$

  where
  - f Hölder
  - μ a T-invariant equilibrium measure
- new:
  - non-stationary/sequential dynamics $\Longrightarrow$ model non-equilibrium and time-varying systems
  - need not have an invariant measure, so use a reference measure (e.g. Lebesgue)
  - in some settings the CLT was already proven, obtain now the ASIP

Almost Sure Invariance Principle

- $X_k : (\Omega, \text{Prob}) \to \mathbb{R}$ random variables
- $S_n = X_1 + \cdots + X_n$
- $\sigma_n^2 := \text{Var}(S_n) \to \infty$

Definition (ASIP)

The sequence S_n satisfies a non-stationary (self-norming) ASIP if

- can find a new (enlarged) probability space (Ω', Prob') with a Brownian motion $W_t : \Omega' \to \mathbb{R}$ on it
- can reproduce in distribution the family S_k on Ω' by S'_k
- for some $\delta > 0$:

$$S'_k - E(S_k) = W(\sigma_k^2) + \mathcal{O}(\sigma_k^{1-\delta}) \qquad a.s.$$

What we obtain (another way to state ASIP)

Definition (Meaning of "ASIP holds for $\{\varphi_\ell(T_\ell \cdots T_1)\}_\ell$")

After enlarging the space, can find a sequence of independent centered normal RV's Z_k such that

$$\sup_{1\le k\le n} \left|\sum_{i=1}^{k} \tilde{\varphi}_i(T_i \cdots T_1) - \sum_{i=1}^{k} Z_i\right| = o(\sigma_n^{1-\beta}) \qquad a.s.$$

where $\sim$ stands for subtracting the mean.
Furthermore $\sum_{j=1}^{n} E[Z_i^2] = \sigma_n^2 + \mathcal{O}(\sigma_n)$.
(Probability measure can be Lebesgue, or the acip of T when $T_i \equiv T$.)

$$Z_k = W(\sigma_k^2) - W(\sigma_{k-1}^2)$$

ASIP $\Longrightarrow$ CLT, LIL

Assume $\{X_j\}_j$ satisfies the ASIP (and have mean zero). Then:

- it satisfies a (self-norming) Central Limit Theorem:

$$\frac{1}{\sigma_n}\sum_{j=1}^{n} X_j \to \mathrm{N}(0,1)$$

  where the convergence is in distribution.

- it satisfies a (self-norming) Law of Iterated Logarithm:

$$\limsup_n \frac{\sum_{j=1}^n X_j}{\sqrt{2\sigma_n^2 \log\log(\sigma_n)}} = 1 \qquad a.s.$$

$$\liminf_n \frac{\sum_{j=1}^n X_j}{\sqrt{2\sigma_n^2 \log\log(\sigma_n)}} = -1 \qquad a.s.$$

Notations, assumptions

- $\mathcal{T}_n := T_n \circ \cdots \circ T_1, \qquad \mathcal{P}_n := P_{\mathcal{T}_n} = P_n \circ \cdots \circ P_1$
- $P_k : L^1(X, m) \to L^1(X, m)$ is the Perron-Frobenius (transfer) operator for $T_k : X \to X$, "predual" of $f \mapsto f \circ T_k$ on $L^\infty(m)$:

$$\int_X P_T f \cdot g \, dm = \int_X f \cdot g \circ T \, dm, \quad \text{for all } f \in L^1(X, m),\ g \in L^\infty(X, m)$$

  [T must be non-singular wrt m, i.e., $m(A) \neq 0 \implies m(T(A)) \neq 0$]
- $\mathcal{V} \subset L^1$ can be BV or Hölder functions (so that P_T is quasi-compact on $\mathcal{V}$, $\|\varphi\|_\infty \leq C\|\varphi\|_\mathcal{V}$ on $\mathcal{V}$)
- **Property (DEC)** of a family $\mathcal{F}$ of operators [Conze & Raugi]: there are $C > 0, \gamma \in (0, 1)$, such that for any n and any sequence $P_k \in \mathcal{F}$ and any $f \in \mathcal{V}$ of zero mean

$$\|P_n \circ \cdots \circ P_1 f\|_\mathcal{V} \leq C\gamma^n \|f\|_\mathcal{V}$$

- **Property (MIN)** of the family $\mathcal{F}$ [Conze & Raugi]: there is $\delta > 0$ such that for any sequence P_k in $\mathcal{F}$

$$\inf_x P_n \circ \cdots \circ P_1 \mathbf{1}(x) \geq \delta, \quad \forall n \geq 1.$$

ASIP for expanding maps on the interval

Theorem

- (φ_n) *a sequence in* $\mathcal{V}$ *such that* $\sup_n \|\varphi_n\|_{\mathcal{V}} < \infty$.
- *(DEC) and (MIN) hold*
- $\sigma_n^2 = \mathrm{Var}(\sum_{j=1}^n \varphi_j \circ \mathcal{T}_j) \geq n^{1/2+\delta}$ *for* $0 < \delta$

Then $\{\varphi_n \circ \mathcal{T}_n\}_n$ *satisfies the ASIP.*

E.g., can take

- β-transformations, $\mathcal{V}$ =BV: $T_n(x) = \beta_n x$ (*mod* 1) on the unit interval with β_n approaching a fixed $\beta \in (1, \infty)$, and $\varphi_n \equiv \varphi$ not a coboundary for T_β (Conze & Raugi)
- piecewise expanding maps in higher dimension, $\mathcal{V}$ = quasi-Hölder functions (similar to Saussol's setting)

- situation considered by Nándori, Szász and Varjú, *A CLT for time-dependent dynamical systems* (2006), provided σ_n grows as needed. E.g.:
  - $T_n(x) = a_n x \ (mod\ 1)$, $a_n \geq 2$ integer,
  - $\varphi_n \equiv \varphi \in \mathcal{V} = C^1(S^1)$, $\int \varphi \, d\ m = 0$, and
    (1) either same value b is repeated for longer and longer stretches in the $a_n's$, and φ is not a coboundary for T_b
    (2) or there are infinitely many k's for which $\min\{a_k, a_{k+1}, a_{k+2}\} > L$, for L large determined by φ.

In either of these situations $\sigma_n \to \infty$ and a self-norming CLT holds.

Can have in (1) variance growing arbitrarily slowly (but CLT holds).

ASIP for the shrinking target problem, expanding maps

Theorem

- *$X = [0,1]$, $T : X \to X$ expanding map, preserves acip μ with density bounded below*
- *P_T has exponential decay in BV (P_T defined wrt μ, so $P_T \mathbf{1} = \mathbf{1}$)*
- *$A_n \subset X$ sequence of nested sets, $\varphi_n := \mathbf{1}_{A_n}$ such that:*
  - $\sup_n \|\varphi_n\|_{BV} < \infty$
  - $\mu(A_n) \geq n^{-\gamma}$ *for some* $\gamma > 0$
  - $E_n := \sum_{j=1}^{n} \mu(A_n)$ *diverges.*

Then $\{\varphi_n \circ T^n\}_{n \geq 1}$ satisfies the ASIP.

E.g.: β-transformations, smooth expanding maps, the Gauss map, mixing Rychlik-type maps

ASIP for non-stationary observations on invertible hyperbolic systems.

Theorem

- *(T, X, μ) an Axiom-A dynamical system preserving the Gibbs measure μ*
- *$\varphi_j \in C^\alpha$ a sequence of α-Hölder functions*
- $\sup_j \|\varphi_j\|_\alpha < \infty$
- $\sigma_n^2 = \mathrm{Var}_\mu(\sum_{j=1}^n \varphi_j \circ T^j) \geq Cn^\delta$ *for some* $\delta > \frac{\sqrt{17}-1}{4} \approx 0.78$

Then $\{\varphi_j \circ T^j\}_j$ satisfies the ASIP.

Follows from a similar result for an expanding map T and Hölder observations φ_n, by coding with a 2-sided shift and reducing to the future coordinates.

Reverse martingale $\Longrightarrow$ ASIP

Theorem (Cuny & Merlevède, 2014)

- *reverse martingale:*
  $\searrow$ *filtration* $(\mathcal{G}_n)_{n\geq 1}$, $X_n \in L^2(\mathcal{G}_n, P)$, $E(X_n|\mathcal{G}_{n+1}) = 0$ *a.s.*
- $\sigma_n^2 := \sum_{k=1}^n E(X_k^2) \to \infty$, $\sup_n E(X_n^2) < \infty$
- $0 < a_n \nearrow$ *such that* $(a_n/\sigma_n^2)_{n\in N} \searrow$ *and* $(a_n/\sigma_n)_{n\in N} \nearrow$

Assume that

(A) $\sum_{k=1}^n \left[E(X_k^2|\mathcal{G}_{k+1}) - E(X_k^2)\right] = o(a_n) \qquad P\text{-a.s.}$

(B) $\sum_{n\geq 1} a_n^{-v} E(|X_n|^{2v}) < \infty$ *for some* $1 \leq v \leq 2$

Then (after enlarging the space): can find $(Z_k)_{k\geq 1}$ *independent centered normal RV's with* $E(Z_k^2) = E(X_k^2)$ *and*

$$\sup_{1\leq k\leq n} \left|\sum_{i=1}^k X_i - \sum_{i=1}^k Z_i\right| = o\Big(\big(a_n(|\log(\sigma_n^2/a_n)| + \log\log a_n)\big)^{1/2}\Big) \qquad P\text{-a.s.}$$

Reverse martingale (follow [Conze & Raugi])

- $\mathcal{T}_n := T_n \circ \cdots \circ T_1, \quad \mathcal{P}_n := P_n P_{n-1} \cdots P_1, \quad Q_n \varphi := \frac{P_n(\varphi \mathcal{P}_{n-1} \mathbf{1})}{\mathcal{P}_n \mathbf{1}}$
- $\mathcal{B}_n := \mathcal{T}_n^{-1} \mathcal{B}$ the $\searrow$ filtration
- then $Q_n T_n \varphi = \varphi, \qquad E[\varphi \circ \mathcal{T}_k | \mathcal{B}_k] = (Q_k \varphi)(\mathcal{T}_k)$
- denote $\tilde{\varphi}_n := \varphi_n - m(\varphi(T_n \cdots T_1))$ and let

$$h_n := Q_n \tilde{\varphi}_{n-1} + Q_n Q_{n-1} \tilde{\varphi}_{n-2} + \cdots + Q_n Q_{n-1} \cdots Q_1 \tilde{\varphi}_0$$

$$\psi_n := \tilde{\varphi}_n + h_n - T_{n+1} h_{n+1}$$

- then $U_n := \psi_n \circ \mathcal{T}_n$ is a reverse martingale, and

$$\sum_{k=1}^{n} \psi_k(\mathcal{T}_k) = \sum_{k=1}^{n} \tilde{\varphi}_k(\mathcal{T}_k) + h_1(\mathcal{T}_1) - h_n(\mathcal{T}_{n+1})$$

- (DEC) and (MIN) $\implies \sup \|h_n\|_{\mathcal{V}} < \infty$

ASIP for expanding interval maps (Thm. 1)

- [Conze & Raugi] prove, for $U_n := \psi_n \circ \mathcal{T}_n$

$$\int [\sum_{k=1}^{n} E(U_k^2|\mathcal{B}_{k+1}) - E(U_k^2)]^2 \, dm \leq c_1 \sum_{k=1}^{n} E(U_k^2) \leq c_2 \sigma_n^2$$

- take $a_n = \sigma_n^{2-\varepsilon}$
- Use Gál-Koksma:

$$\sum_{k=1}^{n} \left[E(U_k^2|\mathcal{B}_{k+1}) - E(U_k^2) \right] = o(\sigma_n^{1+\eta}) = o(a_n) \qquad m\text{-a.s.}$$

  so Condition (A) of the Cuny & Merlevéde Theorem
- take $v = 2$ in Condition (B)
- $\implies \{U_n = \psi_n(\mathcal{T}_n)\}_n$ satisfies the ASIP, and

$$\sum_{k=1}^{n} \psi_k(\mathcal{T}_k) = \sum_{k=1}^{n} \tilde{\varphi}_k(\mathcal{T}_k) + h_1(\mathcal{T}_1) - h_n(\mathcal{T}_{n+1})$$