Computations of singular solutions of an optimal transport problem

Mike Cullen Met Office

Colin Cotter and Abeed Visram (Imperial College)

Contents

This presentation covers the following areas

- Background
- Problem formulation
- Direct solution using optimal transport
- Solution using ‘standard’ numerical methods

Background

Governing equations

On all relevant scales, the atmosphere is governed by the compressible Navier-Stokes equations, the laws of thermodynamics, phase changes and source terms

The solutions of these equations are very complicated, reflecting the complex nature of observed flows

The accurate solution of these equations would require computers 10^{30} times faster than now available

Therefore cannot guarantee that numerical model solutions will be useful

Uses of reduced models

Therefore identify reduced models valid in specific asymptotic limits whose solutions are computable.

Use these to validate operational models by testing convergence to the solution of the reduced model.

Ideally there will be rigorous mathematical proofs that such convergence occurs.

Problem formulation

Eady model

The incompressible Euler equations in a 2d domain $(x_1,x_3)\epsilon(-L,L)x(0,H)$, and with D_t , and ∇ operating only in 2d, are:

$$D_t u_1 + \frac{\partial p}{\partial x_1} - f u_2 = 0$$

$$D_t u_2 + f u_1 = fU\left(x_3 - H/2\right)$$

$$D_t u_3 + \frac{\partial p}{\partial x_3} + \rho g = 0$$

$$D_t \rho + g^{-1} f u_2 U = 0$$

$$\nabla \cdot u = 0$$

Eady model scaled

With $\varepsilon = U/fL$, L/H can be O(1). $u_1 = \varepsilon u_2$.

$$\varepsilon^2 D_t u_1 + \frac{\partial p}{\partial x_1} - u_2 = 0$$

$$D_t u_2 + u_1 = U\left(x_3 - \frac{1}{2}\right)$$

$$\varepsilon^2 D_t u_3 + \frac{\partial p}{\partial x_3} + \rho = 0$$

$$D_t \rho + u_2 U = 0$$

$$\nabla \cdot u = 0$$

Semi-geostrophic limit

$$\frac{\partial p}{\partial x_1} - u_2 = \mathrm{O}\left(\varepsilon^2\right)$$

$$D_t u_2 + u_1 = U\left(x_3 - \frac{1}{2}\right)$$

$$\frac{\partial p}{\partial x_3} + \rho = \mathrm{O}\left(\varepsilon^2\right)$$

$$D_t \rho + u_2 U = 0$$

$$\nabla \cdot u = 0$$

Semi-geostrophic limit in dual variables

$$X_1 = x_1 + u_2$$

$$X_3 = -\rho$$

$$P = p + \frac{1}{2} x_1^2$$

$$(X_1, X_3) = \nabla P$$

$$D_t X_1 = U\left(x_3 - \frac{1}{2} \right)$$

$$D_t X_3 + u_2 U = 0$$

$$\nabla \cdot u = 0$$

Direct solution using optimal transport

Solution using optimal transport

Define mass in dual variables.

$$\sigma = \nabla P \# L$$

Define velocity in dual variables

$$D_t\left(X_1, X_3\right) \equiv \mathbf{U} = \left(U\left(x_3 - \frac{1}{2} \right), -u_2 U \right) = U\left(\left(x_3 - \frac{1}{2} \right), (x_1 - X_1) \right)$$

Solve the mass conservation equation in dual variables.

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \left(\sigma \mathbf{U} \right) = 0$$

Map solution back to physical space at each time using optimal transport to calculate dual space velocity. Proof by Benamou and Brenier.

Optimal transport problem

Find map *T* minimising energy *E* where.

$$\left(x_1, x_3\right) = T\left(X_1, X_3\right)$$

$$E = \int \frac{1}{2}\left(x_1 - X_1\right)^2 - x_3 X_3$$

$$T_{\#}\sigma = L$$

Solution satisfies

$$\left(X_1, X_3\right) = \nabla P$$

as required

Numerical solution

Represent σ as a set of Dirac masses σ_i.

Solve optimal map and transport masses with velocity $\mathbf{U}_i$.

Proved to converge to correct solution by Cullen, Gangbo and Pisante.

Solution using ‘standard’ numerical methods

Issues

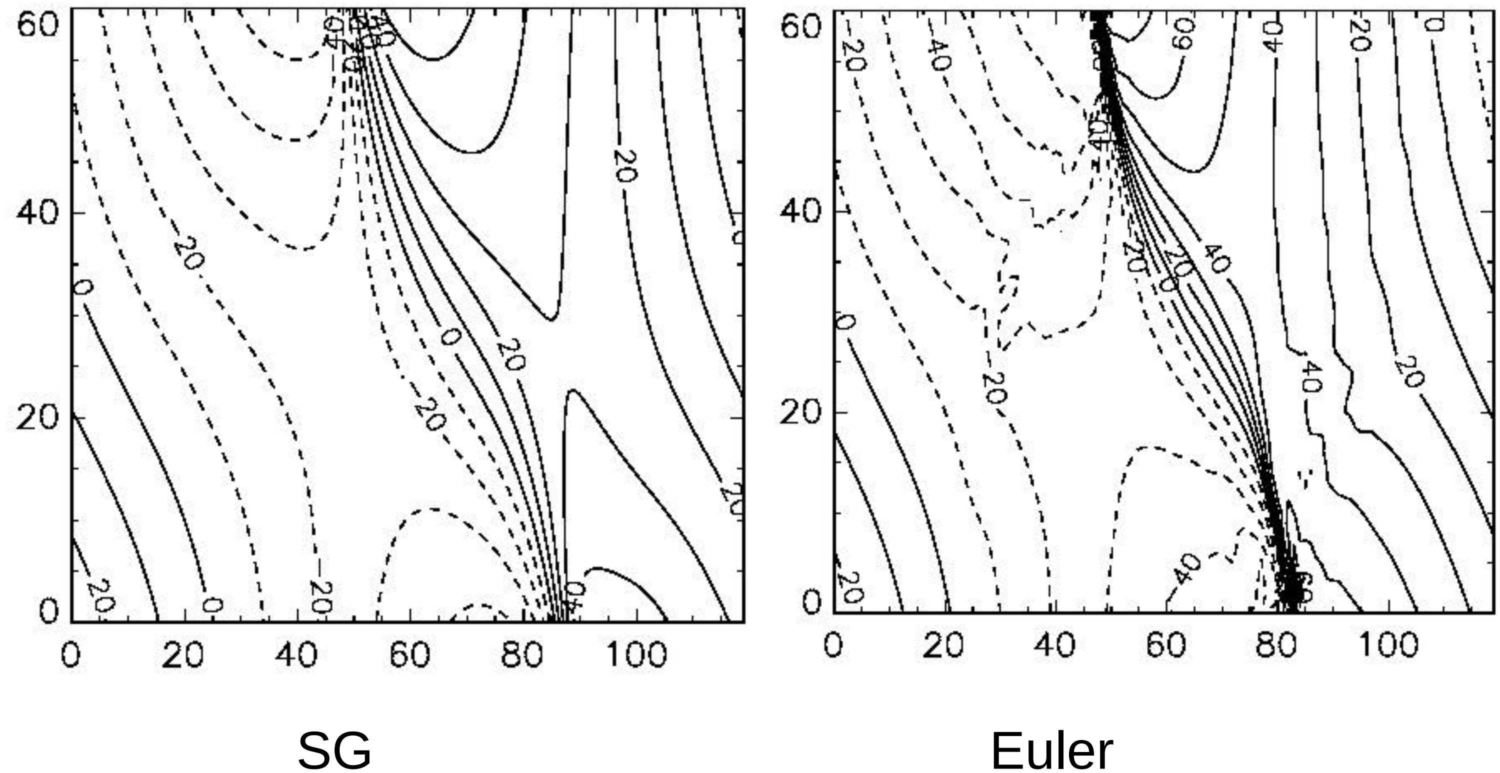
Method only works well in 2 dimensions due to inaccurate discretisation of evolution equation.

Seek more general (thus Eulerian) method.

Direct solution of physical space equations using standard numerical methods does not work as well as solutions of Euler equations using standard numerical methods and extrapolating to small ε.

Existence of SG solution in physical space only proved for weak Lagrangian solutions (Cullen and Feldman)-so Eulerian discretisation is problematic.

Illustration



Discretisation of physical space equations built into direct solution procedure cannot be very good. Better schemes can be built into Euler.

Solving Euler is better conditioned and easier to solve, can be viewed as an iteration towards SG.

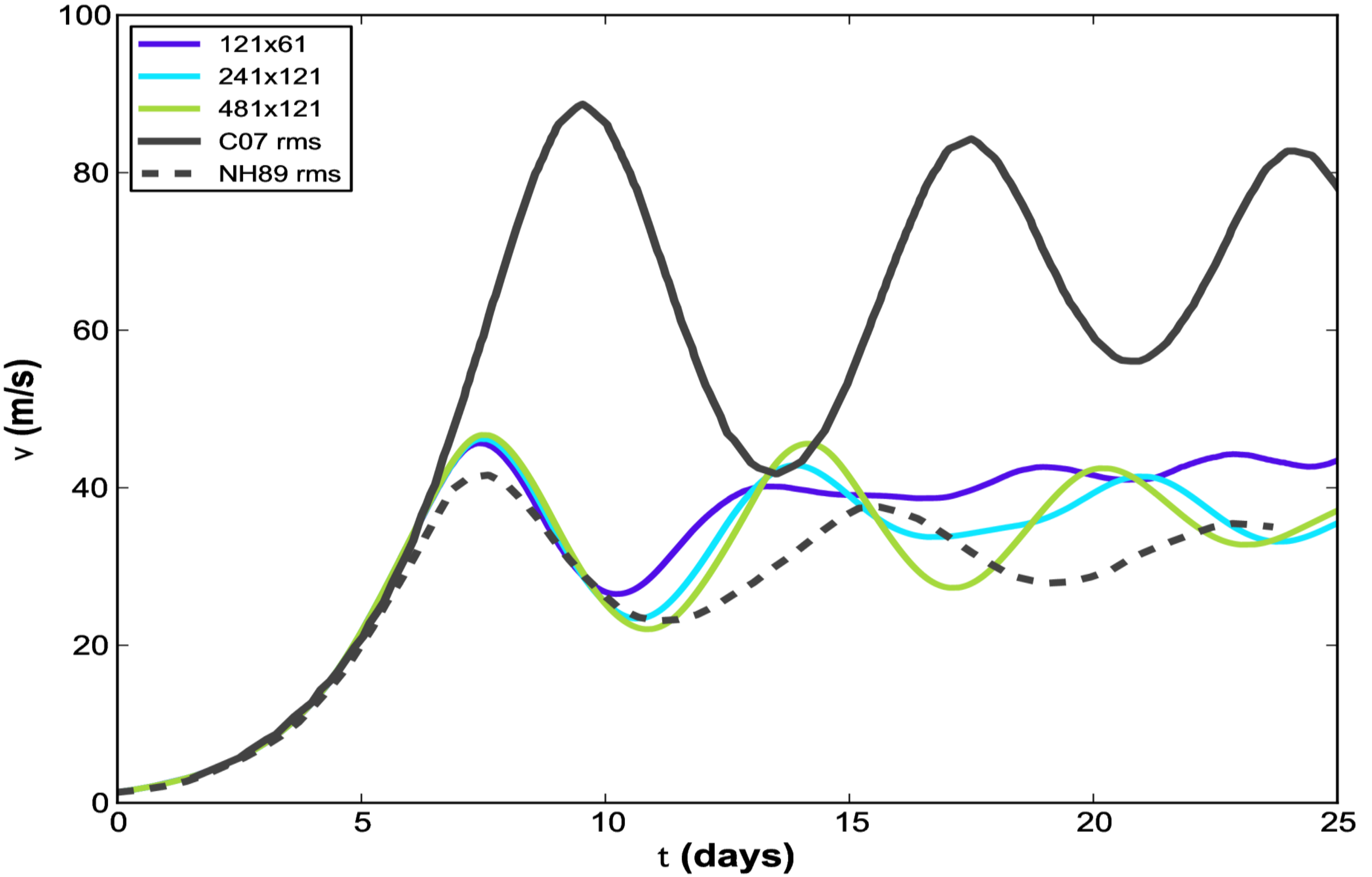
Issues with iteration method

Convergence of Euler to SG proved in smooth case by Brenier and Cullen

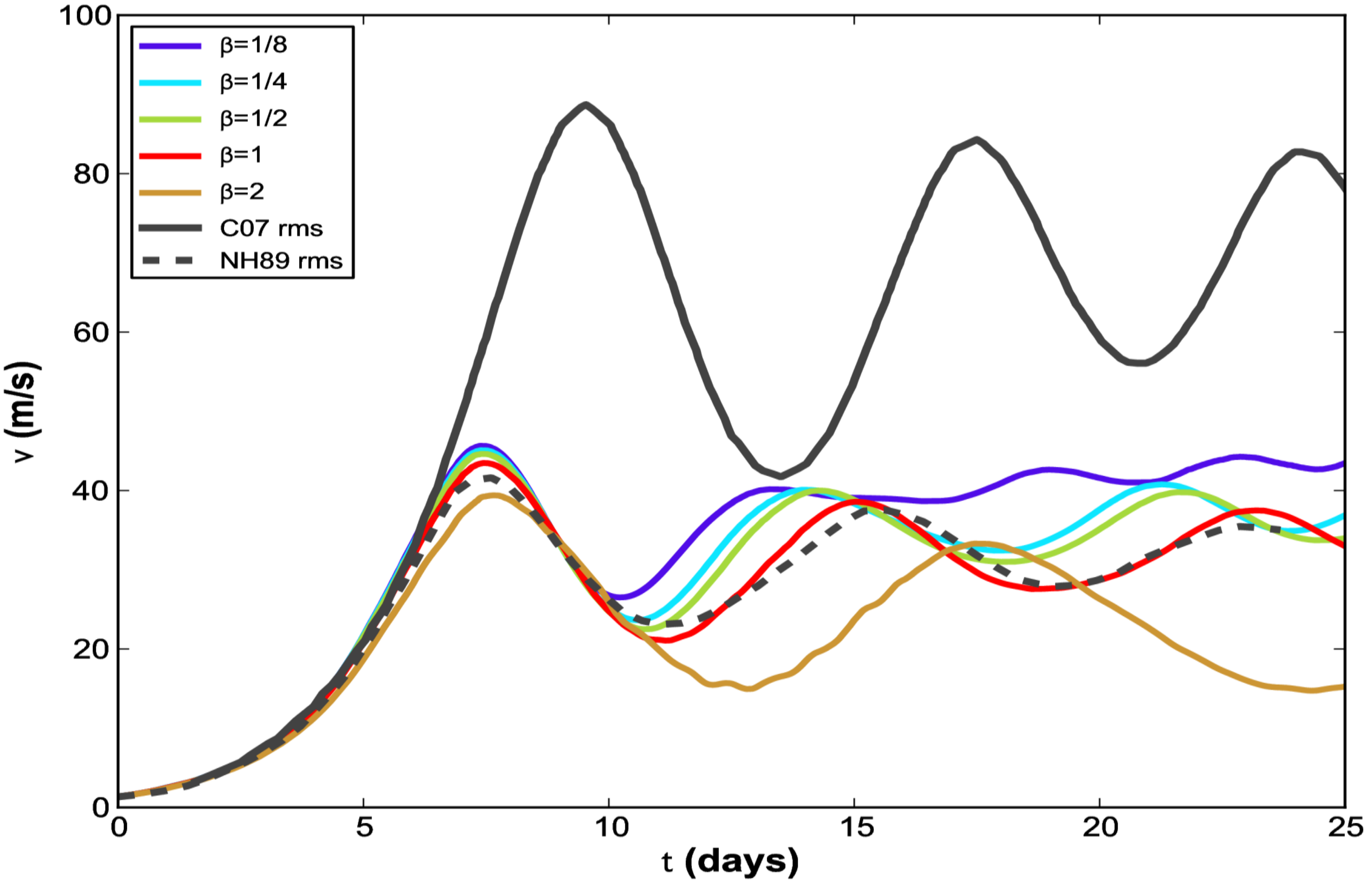
Demonstration of convergence by calculating $O(\varepsilon^2)$ terms from Euler solution.

Issue is Lagrangian conservation. SG solution determined by this property which is also satisfied by Euler equations.

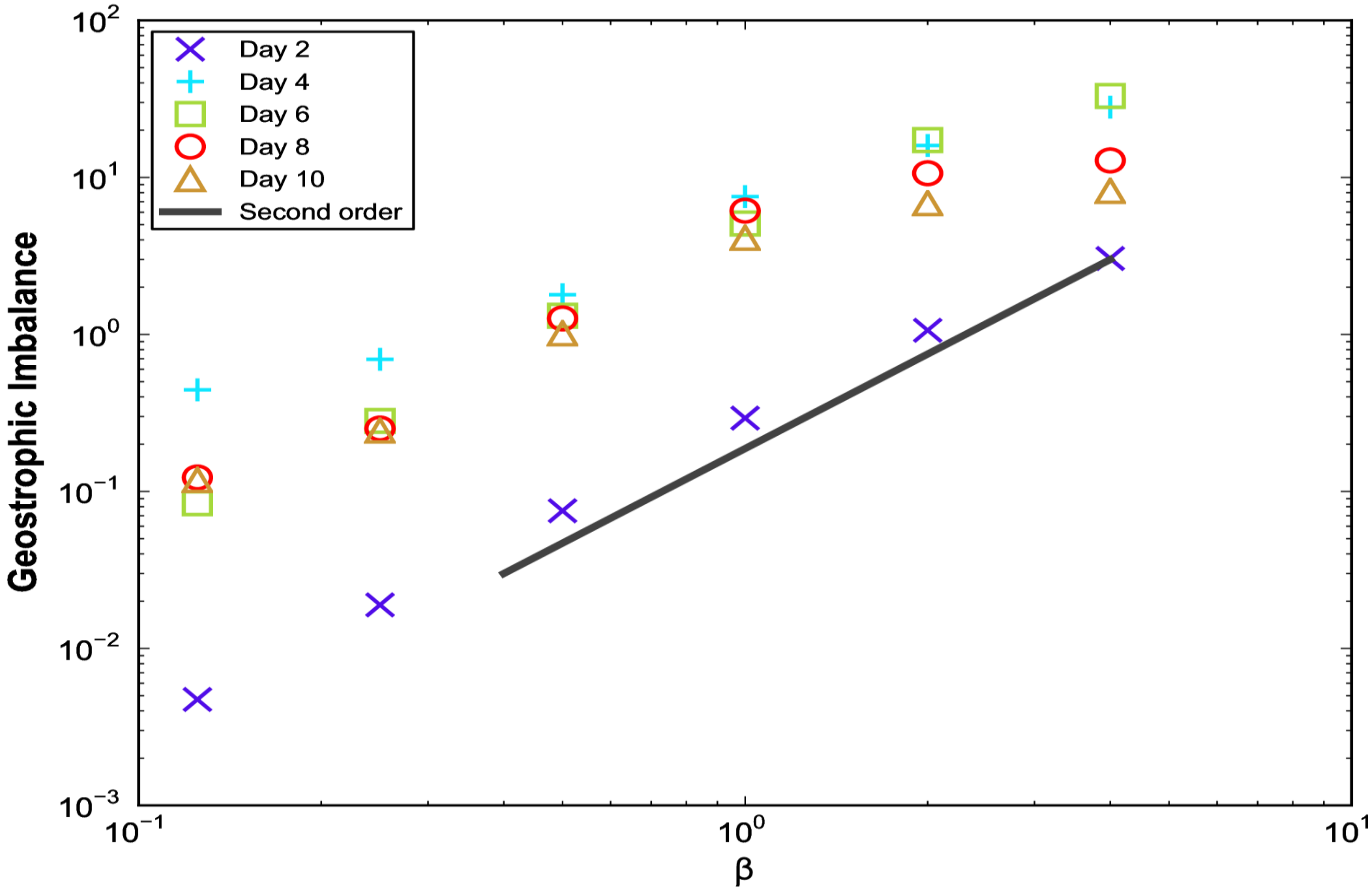
Compare direct method with Eulerian-different resolutions



Convergence to direct SG solution as ε reduced



Convergence to geostrophic balance as ε reduced



Comments

Model gives the correct rate of convergence to SG.

Peak amplitude not predicted. This is because nonlinearity stops the linear growth too quickly.

Implicit diffusion due to the limiters in the advection scheme balances the frontogenesis. Lagrangian conservation under advection is badly violated when solution becomes singular.

Enforcing conservation

Illustrate the difference between $(\rho_a)^2$ and $(\rho^2)_a$ normalised by ρ^2 under advection for one timestep using smooth prefrontal flow fields.

V				ρ			
	1	1.4E-3	1.0E-4		1	2.2E-4	2.2E-5
L_∞	2	5.3E-4	3.0E-5	L_∞	2	1.1E-4	6.6E-6
	4	4.3E-4	9.8E-6		4	3.1E-5	2.1E-6
Conv	rate	0.88	1.70			1.41	1.70
L_2	1	2.9E-5	6.2E-6	L_2	1	6.1E-6	1.8E-6
	2	6.4E-6	9.9E-7		2	1.6E-6	3.2E-7
	4	2.9E-6	2.2E-7		4	4.2E-7	7.2E-8
Conv	rate	1.67	2.41			1.92	2.33

Enforcing conservation

Met Office

Illustrate the difference between $(\rho_a)^2$ and $(\rho^2)_a$ normalised by ρ^2 under advection for one timestep using sharp post frontal flow fields.

V				ρ			
	1	8.7E-2	1.7E-1		1	1.3E-2	1.5E-2
L_∞	2	8.7E-1	9.5E-1	L_∞	2	3.3E-2	3.2E-2
	4	1.9E0	1.9E-0		4	5.4E-2	8.1E-2
Conv	rate	-2.21	-1.70			-1.02	-1.21
L_2	1	3.1E-3	4.3E-3	L_2	1	4.0E-4	3.9E-4
	2	1.4E-2	1.7E-2		2	7.0E-4	7.9E-4
	4	1.3E-2	2.1E-2		4	6.8E-4	4.7E-4
Conv	rate	-1.05	-1.16			-0.07	-0.40

Summary

Direct solution of embedded optimal transport problem using fully Lagrangian method works, but limited applicability.

Direct solution not very good using general Eulerian methods when solution is singular.

Relaxed solution using Euler equations and taking limit is better, but Lagrangian conservation still needs improvement.

Questions