Advances in Numerical Optimal Transportation

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1 Introduction

Optimal Transportation (OT) is a mathematical research topic which began two centuries ago the French mathematician Monge's work on "des remblais et déblais" in 1781. This engineering problem consists in minimizing the transport cost between two given mass densities. In the 40's, Kantorovitch solved the dual problem and interpreted it as an economic equilibrium. The *Monge-Kantorovitch* problem became a specific research topic in optimization and Kantorovitch obtained the 1975 Nobel prize in economics for his contributions to resource allocation problems. Following the seminal discoveries of Brenier in the 90's, Optimal Transportation has received renewed attention from mathematical analysts, resulting in the Fields Medal awarded in 2010 to C. Villani, who gave important contributions to Optimal Transportation, arrived at a culminating moment for this theory. *Optimal Transportation* is today a mature area of mathematical analysis. Numerical methods and applications are comparatively underdeveloped. The workshop was dedicated to recent results and methods pertaining to modelization and numerical simulations using Optimal Transportation tools and concepts.

2 Numerical approaches to Optimal Transportation

The Augmented Lagrangian method applied to the CFD Optimal Transportation [31] and its variants has been for a long time the main tool in numerical optimal transportation. It was used in particular for warping and registration in image processing. Three other numerical approaches have recently been explored :

Solving the Monge-Kantorovich linear program : Cuturi [8, 9, 10, 11, 4] presented the idea of regularizing optimal transport problems with an entropic penalty to enforce desirable properties for optimal couplings, such as balanced flows between equally expensive routes and overall smoothness. Only recently was it shown that such a regularization can provide an extremely efficient computational framework to approximate optimal transport using the toolbox of (strict) convex optimization. This presentation was followed by Peyré's with various applications of this family of numerical method to JKO gradient flows [20], and Wasserstein barycenters [4] in particular. For applications in neuroscience see [15] [14].

Ruan Yuanlong also presented a **multiscale linear programming solver** for optimal transportation (without Entropic regularisation).

Laguerre Cells for Semi-Discrete Optimal Transportation : Levy [17] presented a 3D implementation of quadratic OT between a density and a sum of Dirac masses. The modern and implementation relies of the efficient computation of Laguerre Cells and the Newton or Quasi Newton minimisation of a convex objective function. Merigot proposed a hierarchical algorithm that improves the speed of convergence, together with an implementation in 2D [32]. The numerical algorithm has been tested on several datasets, with up to hundred thousands tetrahedra and one million Dirac masses.

Finite difference Monge Ampère solvers and discretization of the cone of convex function : Mirebeau reviewed the different approaches to discretize optimisation of functionals under convexity constraint [35]. This was illustrated on the Principal Agent problem in 3D. He then explained how it can be use to solve the Monge-Ampère equation [34] [33]. This is the only monotone FD scheme for Optimal Transportation after [40, 41].

3 Extensions

Optimal Transportaion on graphs : Qinglan Xia [42] also presented his numerical simulations approach of the ramified optimal transportation.

Multi-Marginal Optimal Transport : Multi-Marginal optimal transportation is a new concept extending Optimal Transportation when data consists in more than 2 densities. Pass [19] gave a general overview of the multi-marginal optimal transport problem and outlined several applications.

4 JKO gradient flows and fluids, new modelisation and numerical simulations

Minimizing the Wasserstein distance to between a source measure and a target measure can be regarded as an infinite dimensional two point boundary problem. The mapping between the two densities also generates, by interpolation, a geodesic in the Wasserstein distance between the two measures (regarded as points).

A different extension is to consider a gradient flow in the Wasserstein metric, in other words, to evolve the density in order to minimize some functional. Computing the steepest descent direction with respect to the Wasserstein distance defines a semi discrete Wasserstein gradient flow, also known as a *JKO gradient flow* [30]. It is now well-known, that in the limit, several diffusion and aggregation equations of second and fourth order can be interpreted as gradient flows of an entropy in the L^2-Wasserstein metric. The theory is well developed [3] and continues to expand, see for instance new convergence results in [22]. An significant part of the workshop was devoted to new theoretical and numerical results in this field.

JKO schemes Theory : Santambrogio presented some new L^∞ estimates obtained in collaboration with J.-A. Carrillo. for the Keller-Segel model of chemiotaxis, based on a JKO-like scheme, and on a fine analysis of the optimality conditions at every time step, combined with the use of the Monge-Ampère equation.

Carlier and Agueh [1] presented a new JKO approach to the analysis of kinetic models of granular materials.

JKO schemes Numerical methods : The JKO scheme is semi-discrete in time. These works (try to) address the space discretisation and the convergence of the resulting shemes.

Wolfram [6] presented a finite element methods for two optimal transportation problems, in particular a class of nonlinear convection-aggregation equations and the Monge-Ampere equation which discretize the associated JKO scheme. OT provides a Lagrangian change of variable into the non-linear PDE to be solved.

Osberger and Mathes [36] use the Lagrangian change of variable into the JKO scheme and perform the gradient flow descent on a galerkin approximation of the transport map. The mass densities can be viewed as weighted moving particles. They check convergence.

Pataccini also uses the Lagrangian approach but chooses to represent the densities as balls of various size along particles. Gamma convergence results are available. This is joint work with J. A. Carrillo, Y. Huang, P. Sternberg and G. Wolansky.

See also Peyré contribution in section 2.

5 Applications

Fluids Dynamics : Maury presented an optimal transportation framework to pressureless Euler equation with a maximal density constraint. This is a second order extension of the the macroscopic congested crowd motion models proposed in [37].

Cullen [29, 7] presented and discussed computations of singular solutions of the semi-geostrophic eady problem using solutions of the Euler problem. The Semi-Geostrophic problem is the asymptotic limit as the Rossby number tends to zero.

Mérigot [18] presented a numerical method to solve Brenier Generalized Euler solutions in 2D which is an instance of Multi-Marginal optimal transportation.

Seismology : In seismic exploration a wave field is generated at the surface and reflections from the earths interior are recorded. The purpose is to find properties such as wave velocity and location of reflecting sub layers. This is done in an inverse process where the measurements are compared to a computed wave field with unknown coefficients in the wave equation. Engquist and Froese [13] propose to use the Wasserstein metric for this comparison. We will also remark on applications to registration in seismology.

Non-Imaging optics : Free-Forming or the automatic design of smooth refractors or reflectors can be modelled with Monge-Ampère type PDEs and optimal transportation.

Gutierrez [16] gave a complete mathematical overview of the refractor problem : design of interface surfaces between the two homogeneous materials, so the resulting lense refracts radiation in a prescribed manner.

The far-field reflector problem is also a well-known inverse problem arising in geometric optics. It consists in creating a mirror that reflects a given point light source to a prescribed target light at infinity. Thibert [12] presented how the combinatorics of this intersection can be efficiently computed using tools from computational geometry, thus providing an efficient algorithm for the far field reflector problem. This work is in common with Pedro Machado et Quentin Mérigot. [12]

Thije Boonkkamp [21] also addressed the reflector problem but using finite difference based discretisation of the reflector and in a far field regime.

Image Processing : Solomon [27, 28, 26, 25, 24, 23] reviewed several transportation techniques for Geometric Data Processing. Mainly based on accelerated LP solvers and entropic regularization.

Hongkai Zhao [39] uses a simplification of Wasserstein distance called Sliced-Wasserstein Distance to perform Multi-scale Non-Rigid Point Cloud Registration. He applies the Sliced-Wasserstein Distance density buit with the Laplace-Beltrami Eigenmap Operator.

Saumier [38] presented a method based on Optimal Optimal Transport for Particle Image Velocimetry.

Mesh Adaptation : Budd [5] borrows idea from optimal transport to construct a moving mesh adapted to a transient solution. He uses a parabolic Monge-Ampère equation to compute an approximation of the optimal map at each time step. This is applied to the Semi-Geostrophic problem mentioned above.

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