Classifications of parabolic germs and fractal properties of orbits

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The problem considered

Germs of analytic *parabolic* diffeomorphisms $f:(\mathbb{C},0)\to(\mathbb{C},0)$:

$f(z)=z+a_1z^{k+1}+a_2z^{k+2}+\ldots+a_kz^{2k+1}+o(z^{2k+1}),\ a_1\neq 0,\ k\geq 1.$

- ⋆ k... the *multiplicity* of fixed point 0
- ⋆ **Leau-Fatou flower theorem** (1987):
  - k attracting directions: $(-a_1)^{-\frac{1}{k}}$; k repelling directions: $a_1^{-\frac{1}{k}}$

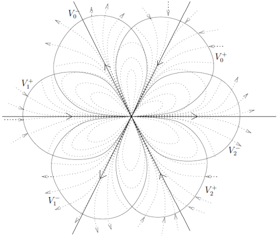


Figure: $f(z)=z+z^4+o(z^4)$

The problem considered

Can we recognize a germ using fractal properties of only one orbit?

- ⋆ formal class of a germ
- ⋆ analytic class of a germ

Fractal properties of a set $U \subset \mathbb{C}$ $(\mathbb{R}^2)$

THE BOX DIMENSION OF A SET (fractal dimension)

- $U \subset \mathbb{R}^2$ bounded
- $\varepsilon > 0$, $|U_\varepsilon|$ the area of the ε-neighborhood
- For $s \in [0, 2]$, we consider

$$\lim_{\varepsilon \to 0} \frac{|U_\varepsilon|}{\varepsilon^{N-s}} \in [0, \infty],$$

and draw:

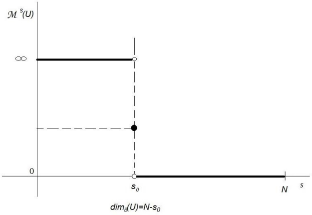


Figure: $s \mapsto \lim_{\varepsilon \to 0} \frac{|U_\varepsilon|}{\varepsilon^{2-s}}$, $s \in [0, 2]$.

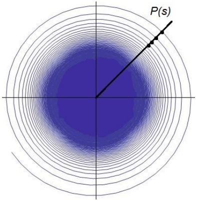
- the moment of jump $s_0 \equiv$ the box dimension, $\dim_B(U) = s_0$.
- the value at $s_0 \equiv$ Minkowski content, $\mathcal{M}(U)$.
- if $|U_\varepsilon| \sim C\varepsilon^{2-s} \;\Rightarrow\; \dim_B(U) = s,\; \mathcal{M}(U) = C$.

Motivation: dynamical systems

Box dimension of spiral trajectory locally around singular point reveals complexity in bifurcations of singular point!

- polynomial vector field, focus point at the origin

$$\begin{cases} \dot{x} &= -y + p(x,y), \\ \dot{y} &= x + q(x,y). \end{cases}$$



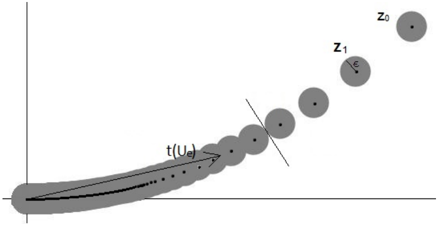
- The Poincaré map $P(s) = s - s^{2k+1} + o(s^{2k+1})$, $k \in \mathbb{N}_0$; focus of order k
- cyclicity in generic bifurcations: k
- $\dim_B(S(x_0)) = \frac{4k}{2k+1}$

(Zubrinić, Županović, *Fractal analysis of spiral trajectories of some planar vector fields* (2005))

Definition (The DIRECTED area of the ε-neighborhood, R)

$$A^{\mathbb{C}}(U_\varepsilon) = |U_\varepsilon| \cdot t(U_\varepsilon) \in \mathbb{C},$$

$t(U_\varepsilon) \in \mathbb{C}$ the *center of mass* of U_ε.



Formal classification of parabolic diffeomorphisms

(Birkhoff, Ècalle, Kimura, ~ 1950)

$\star$ formal changes of variables:

1. $\phi_1(z) = c_1 z,$
2. $\phi_i(z) = z + c_i z^i,\ c_i \in \mathbb{C}, \quad i = 2, 3, \ldots$

$$\widehat{\phi}(z) = \ldots \circ \phi_2^{-1} \circ \phi_1^{-1}(z) = \sum_{l=1}^{\infty} d_k z^k \quad \text{(formal series)}$$

$\Rightarrow$ **formal normal form**

$$f_0(z) = \widehat{\phi} \circ f \circ \widehat{\phi}^{-1}(z) = Exp\left(\frac{z^{k+1}}{1 + \frac{\lambda}{2\pi i} z^k} \frac{d}{dz}\right), \quad \lambda \in \mathbb{C},$$

$$\widetilde{f_0}(z) = z + z^{k+1} + \left(\frac{k+1}{2} - \frac{\lambda}{2\pi i}\right) z^{2k+1}.$$

$\star$ formal type: $(k, \lambda),\ k \in \mathbb{N},\ \lambda \in \mathbb{C}$.

Asymptotic expansion of the directed area

$z_0 \in V_+$ (attracting petal);

Theorem (R, 2013)

$$A^{\mathbb{C}}(\varepsilon, z_0) = K_1 \varepsilon^{1+\frac{2}{k+1}} + K_2 \varepsilon^{1+\frac{3}{k+1}} + \ldots + K_{k-1} \varepsilon^{1+\frac{k}{k+1}} + \\ + H^f(z_0)\varepsilon^2 + K_k \varepsilon^{2+\frac{1}{k+1}} \log \varepsilon + R(z_0, \varepsilon), \qquad (1) \\ R(z_0, \varepsilon) = o(\varepsilon^{2+\frac{1}{k+1}} \log \varepsilon),\ \varepsilon \to 0,$$

$K_i \in \mathbb{C},\ i = 1, \ldots, k+1,$ *independent of* z_0; $H^f(z_0) \in \mathbb{C}$.

Sketch of the proof...

Theorem (R, 2013)

Formal type (k, λ) explicitely from $(k; K_1, K_k)$ in finite expansion of ANY orbit!

Toward analytic classification

Proposition (R)

The mapping

$$f \longmapsto \big(\varepsilon \mapsto A^{\mathbb{C}}(\varepsilon, z_0),\ \varepsilon \in (0, \varepsilon_0)\big)$$

is injective on the set of germs with z_0 in their attracting basin.

BUT

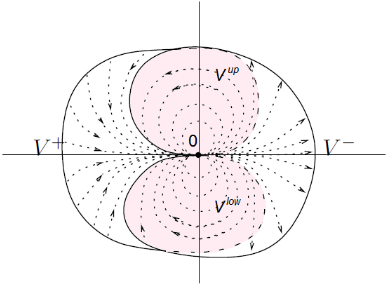
Proposition (R)

No asymptotic expansion of $R(z_0, \varepsilon)$, as $\varepsilon \to 0$, in power-log scale!

The reason. The critical index n_ε a jump function.

The simplest formal class

- ⋆ model formal class $(k = 1, \lambda = 0)$; $f_0 = Exp(z^2 \frac{d}{dz}) = \frac{z}{1-z}$
- ⋆ prenormalized $(a_1 = 1)$
- ⋆ $f(z) = z + z^2 + z^3 + o(z^3)$



Fatou coordinates and moduli of analytic classification

$\Psi(f(z)) - \Psi(z) = 1$ (Abel equation)

Fatou, 1919:

- unique (to a constant) formal solution $\widehat{\Psi}(z) \in -1/z + z\mathbb{C}[[z]]$,
- analytic solutions $\Psi_{\pm}(z)$ on $V_{\pm}$; Gevrey 1-expansion $\widehat{\Psi}(z)$

$\rightarrow$ **Fatou coordinates, sectorial trivialisations**

Ecalle-Voronin moduli of analytic classification

On $V^{up},\ V^{low}$:

$$\Psi_+(f(z)) - \Psi_-(f(z)) = \Psi_+(z) - \Psi_-(z)$$

$\Rightarrow \Psi_+ - \Psi_-$ well-defined on *space of (closed) orbits* of $V^{up},\ V^{low}$

$\rightarrow$ lifts to poles of spaces of orbits of V_+
(spheres, $t = e^{2\pi i\Psi_+}$, orbits $\leftrightarrow$ points):

$$\begin{cases} \Psi_+(z) - \Psi_-(z) & = g_\infty(e^{2\pi i\Psi_+(z)}),\ z \in V^{up}, \\ \Psi_-(z) - \Psi_+(z) & = g_0(e^{-2\pi i\Psi_+(z)}),\ z \in V^{low}. \end{cases}$$

$\rightarrow$ a pair of analytic germs extended to 0,
$t \rightarrow (g_\infty(t), g_0(t))$,
$\rightarrow$ property $g_0(0) + g_\infty(0) = 0$.

Ecalle-Voronin moduli of analytic classification

Ecalle-Voronin modulus of f: (g_∞, g_0), up to identifications:

$$(\star)\begin{cases}(g_1(t), g_2(t)) & \equiv (g_3(t), g_4(t)) \Leftrightarrow \\ & g_3(t) = g_1(t) + a, \ g_4(t) = g_2(t) - a, \\ & g_3(t) = g_1(bt), \ g_4(t) = g_2(t/b), \ a \in \mathbb{C}, \ b \in \mathbb{C}^*.\end{cases}$$

Theorem (Ecalle-Voronin)

analytic classes of germs of the model formal type

$\updownarrow$

all pairs of analytic germs at $t = 0$,

$$(g_1(t), g_2(t)), \ g_1(0) + g_2(0) = 0,$$

up to identifications $(\star)$.

$\star$ analytic class of f_0 *trivial*: $(0, 0)$

Definition

- $z \mapsto H^f(z),\ z \in V_+,$
  the **principal initial point dependent part for** f,
- $z \mapsto H^{f^{-1}}(z),\ z \in V_-,$
  the **principal initial point dependent part for** f^{-1}.

$$A^{\mathbb{C}}(\varepsilon, z) = A^{\mathbb{C}}(\varepsilon, f(z)) + z \cdot \varepsilon^2 \pi,\ \varepsilon \text{ small},$$

$$\overset{expansion}{\Longrightarrow} H^f(z) = H^f(f(z)) + z\pi$$

∗ a *cohomological equation* similar to the Abel equation for f

∗ *Stokes phenomenon*: sectorially analytic solutions?

Cohomological equations

- A **cohomological equation** for f:

$$H(f(z)) - H(z) = g(z),\ g(z) \in \mathbb{C}\{z\},\ g \not\equiv 0.$$

Sectorial solutions of cohomological equations *(Fatou, Loray)*

$g(z) = \alpha_0 + \alpha_1 z + O(z^2)$

- a unique *formal* solution $\widehat{H}(z) \in -\frac{\alpha_0}{z} + \alpha_1 Log(z) + z\mathbb{C}[[z]]$ (without the constant term),
- unique *sectorially analytic* solutions $H_\pm(z)$ on $V_\pm$, with expansion $\widehat{H}(z)$, $z \to 0$

Proof constructive!!!

Sectorial analyticity of principal parts

1-Abel equation for f: $H(f(z)) - H(z) = -z$

$\to$ the sectorial solutions $H_+,\ H_-$

Theorem (R)

- *the principal parts $H^f(z)$ i $H^{f^{-1}}(z)$ analytic on $V_\pm$*
- *explicitely related to solutions $H_\pm(z)$ of 1-Abel equation:*

$$\pi H_+(z) - \frac{\pi}{4} + i\pi^2 = H^f(z), \quad z \in V_+,$$
$$\pi H_-(z) - \frac{\pi}{4} = z - H^{f^{-1}}(z), \quad z \in V_-.$$

'*Global*' principal parts

Existence of global analytic solution H of cohomological equation
$\leftrightarrow H_+ - H_- \equiv 0\ (2\pi i)$ on $V^{up,low}$

1. global analytic solution of Abel equation

$$\Leftrightarrow f = \varphi^{-1} \circ f_0 \circ \varphi, \quad \varphi \in z + z^2\mathbb{C}\{z\}.$$

2. **Theorem (R)**

The 1*-Abel equation has a global analytic solution* $H(z)$
$$\Leftrightarrow \quad f(z) = \varphi^{-1}(e^z \cdot \varphi(z)), \ \ \varphi(z) \in z + z^2\mathbb{C}\{z\}.$$

Germs with global solution to Abel and to 1-Abel equation

- $\mathcal{S} = \{f \mid f = \varphi^{-1}(e^z \cdot \varphi(z)),\ \varphi \in z + z^2\mathbb{C}\{z\}\}$
- $\mathcal{C}_0 = \{f \mid f = \varphi^{-1} \circ f_0 \circ \varphi,\ \varphi \in z + z^2\mathbb{C}\{z\}\}$

Example

1. $f_0(z) = \frac{z}{1-z} \in \mathcal{C}_0 \setminus \mathcal{S}$,

$$H_+(z) - H_-(z) = 2\pi i f_0(e^{-2\pi i \frac{1}{z}}),\ z \in V^{up},$$

$$H_-(z) - H_+(z) = -2\pi i + 2\pi i f_0(e^{2\pi i \cdot \frac{1}{z}}),\ z \in V^{low},$$

2. $f(z) = ze^z \in \mathcal{S} \setminus \mathcal{C}_0$,
3. $f(z) = -Log(2 - e^z) \in \mathcal{S} \cap \mathcal{C}_0$.

The sets $\mathcal{S}$ and $\mathcal{C}_0$ in general position $\Rightarrow$ the differences of sectorial solutions on petal intersections insufficient for determining the analytic class

Example 1 computed using Borel-Laplace transform

- Substitute $\widehat{H}(z) = -\log z + \widehat{R}(z),\ \widehat{R} \in z\mathbb{C}[[z]]$ in eqn:

$$\widehat{R}(f_0(z)) - \widehat{R}(z) = -z + \log \frac{f_0(z)}{z}.$$

- change of variables $w = -1/z$:

$$\widehat{\widetilde{R}}(w+1) - \widehat{\widetilde{R}}(w) = w^{-1} - Log(1 + w^{-1}) = \sum_{k=2}^{\infty} (-1)^k \frac{w^{-k}}{k}$$

$$\in w^{-2}\mathbb{C}\{w^{-1}\}.$$

- $b(w) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} w^{-k}$.
- The formal Borel transform ($\mathcal{B}(w^{-k-1}) = \frac{\xi^k}{k!}$):

$$\mathcal{B}\widehat{\widetilde{R}}(\xi) = \frac{\mathcal{B}b(\xi)}{e^{-\xi} - 1}, \quad \mathcal{B}b(\xi) = \frac{e^{-\xi} + \xi - 1}{\xi}.$$

The Laplace transform

- $f(\xi)$ analytic and exp bdd in direction θ:
$$|f(re^{i\theta})| \leq Ce^{Ar}, \; r > 0, A > 0.$$
- The *Laplace transform of f in direction θ*:
$$\mathcal{L}^\theta f(z) = \int_0^{\infty \cdot e^{i\theta}} f(\xi)e^{-z\xi}d\xi.$$
- $\mathcal{L}^\theta f(z)$ an analytic function on the half-plane $Re(ze^{i\theta}) > A$:

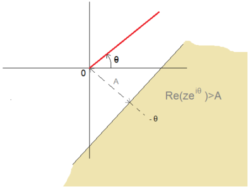


Figure: The direction θ and the corresponding half-plane $Re(ze^{i\theta}) > A$.

...and back by Laplace transform

- $\xi \mapsto \mathcal{B}\widehat{\widetilde{R}}(\xi)$ has 1-poles at $2i\pi\mathbb{Z}^*$ in directions $\pm i$,
- exponentially bounded by $B > 0$ and analytic in other directions
- Laplace transform recovers two analytic solutions:
  $\widetilde{R}^+$ on $W_+ = \{w \,|\, Re(we^{i\theta}) > B,\ \theta \in (-\pi/2, \pi/2)\}$,
  $\widetilde{R}^-$ on $W_- = \{w \,|\, Re(we^{i\theta}) > B,\ \theta \in (\pi/2, 3\pi/2)\}$.
- The Residue theorem: for $w \in W^{up} = \{w \,|\, Im(w) > B\}$

$$\widetilde{R}^+(w) - \widetilde{R}^-(w) = \int_0^{\infty\cdot e^{i\theta_1}} \frac{e^{-\xi w}\mathcal{B}b(\xi)}{e^{-\xi} - 1} d\xi - \int_0^{\infty\cdot e^{i\theta_2}} \frac{e^{-\xi w}\mathcal{B}b(\xi)}{e^{-\xi} - 1} d\xi =$$

$$= -2\pi i \cdot \sum_{k=1}^{\infty} Res(\frac{e^{-\xi w}\mathcal{B}b(\xi)}{e^{-\xi} - 1}, \xi = -2\pi ik) = 2\pi i \frac{e^{2\pi i\cdot w}}{1 - e^{2\pi i\cdot w}}.$$

$\theta_1 \in (-\pi/2, \pi/2)$ and $\theta_2 \in (\pi/2, 3\pi/2)$ close to $-\pi/2$.

- For $w \in W^{low} = \{w \,|\, Im(w) < -B\}$, we get

$$\widetilde{R}^{+}(w) - \widetilde{R}^{-}(w) = -2\pi i \frac{e^{-2\pi i \cdot w}}{1 - e^{-2\pi i \cdot w}}.$$

- Returning to $z = -1/w$ and to $H(z)$, we get

$$H_{+}(z) - H_{-}(z) = 2\pi i \frac{e^{-2\pi i \frac{1}{z}}}{1 - e^{-2\pi i \frac{1}{z}}} = 2\pi i f_0(e^{-2\pi i \frac{1}{z}}), \ \ z \in V^{up},$$

$$H_{-}(z) - H_{+}(z) = 2\pi i + 2\pi i \frac{e^{2\pi i \cdot \frac{1}{z}}}{1 - e^{2\pi i \cdot \frac{1}{z}}} = 2\pi i + 2\pi i f_0(e^{2\pi i \cdot \frac{1}{z}}), z \in V^{low}.$$

□

Classifications of germs with respect to 1-Abel equation

$$H(f(z)) - H(z) = -z$$

$\Rightarrow (H_+ - H_-)(z) = (H_+ - H_-)(f(z)),\ z \in V^{up} \cup V^{low}$

$\Rightarrow H_+ - H_-$ *constant along orbits*

$$H_+ - H_- = g_\infty(e^{2\pi i \Psi_+(z)}),\ \ z \in V^{up},$$
$$H_- - H_+ = -2\pi i + g_0(e^{-2\pi i \Psi_+(z)}),\ \ z \in V^{low}.$$

$\Rightarrow (g_\infty(t), g_0(t)),\ g_\infty(0) + g_0(0) = 0$ a pair of analytic germs

Definition (R)

- The 1-**moment of** f: the pair (g_∞, g_0), up to identifications
- 1-**conjugacy class of** f: $[f]_1$

1-conjugacy classes vs. analytic classes

Theorem (Realization of 1-moments. Transversality.)

(g_0, g_∞) *a pair of analytic germs s.t.* $g_0(0) + g_\infty(0) = 0$. *Then:*

- *There exists a germ in the model formal class such that the given pair is its* 1*-moment.*
- *Moreover, such germ exists inside ANY analytic class.*

Thank you for the attention!