Perspectives on Parabolic Points in Holomorphic Dynamics

Arnaud Chéritat (CNRS/Institut de Mathématiques de Bordeaux, France),
Adam Lawrence Epstein (Math. Institute, Warwick University, UK),
Carsten Lunde Petersen (IMFUFA, Roskilde University, Denmark)

March 29th – April 3rd 2015

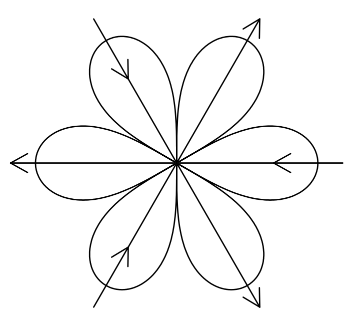
1 History and recent developments

1.1 Parabolic points, classification

One dimensional holomorphic dynamics is concerned with the iteration of holomorphic functions on $\mathbb{C}$ or more generally on a collection of one or more Riemann surfaces.

If a holomorphic map f has a periodic point z, i.e. $f^p(z) = z$, this point is called *parabolic* if the multiplier, i.e. $(f^p)'(z)$, is a root of unity. The classification of parabolic points, up to local conjugacy (by a continuous or a holomorphic map, or by formal power series) has been carried out mainly by Leau, Fatou, Camacho, Écalle, Voronin, Ramis and Martinet.

The Leau-Fatou theorem, [28, 24] describes the dynamics near a parabolic point. There is some $q > 1$ such that a neighborhood can be covered by q attracting *petals* and q repelling petals, alternating, on which the dynamics is conjugate to "the translation by 1 on the right half-plane $\Re z > 0$" for attracting petals, with right replaced by left for repelling petals. Such conjugacies are called *Fatou coordinates*. The number $q + 1$ is equal to the order of tangency between identity and a sufficiently high iterates of f^p. Since the conjugacy can be taken holomorphic, this implies that the quotient of a petal by f^p is isomorphic, as a Riemann surface, to $\mathbb{C}/\mathbb{Z}$.



The classification up to homeomorphism is thus determined by the number q. The formal classification, depends on the number of petals and on a complex number called the formal invariant: $A \in \mathbb{C}$. Each germ can be conjugated by a formal power series to a unique $\rho \times z \times (1 + z^q + Az^{2q})$ with $\rho = (f^p)'(0)$. The classification up to a holomorphic change of variable involves a more complicated invariant, which has been expressed in analytic terms by Écalle (coefficients appearing in alien derivations associated to a resurgence phenomenon of the Borel resummation of asymptotic expansion of Fatou coordinates) and in terms of conformal geometry by Voronin: the change of variables between repelling and attracting Fatou coordinates defines a holomorphic function, the *horn map* on the upper/lower half-plane that commutes with $z + 1$ (they thus in fact live in the cylinders), and provides conjugacy invariants, the so called Voronin invariants. The link between the two points of view is that Écalle's coefficients are nothing but the Fourier coefficients of Voronins horn maps.

The approach by Écalle is rich but extremely complicated, and 30 years after it was proposed, its elucidation is still under way.

Practical ways to obtain estimates on the Fatou coordinates, the horn maps or the Écalle invariants, have been studied by many researchers, including Lanford-Yampolsky in their study of parabolic renormalisation, L.-Y. and Chéritat in the production of computer pictures, and Sylvain Bouillot for Écalle's invariants.

Let us also cite extension toward higher dimension: dynamics near fixed points tangent to identity and its classification has been studied by Écalle, Hakim, Abate, Vivas and Rong, among others. See [2] for references and an exhaustive survey or [3], pages 38–49 for a shorter one.

1.2 Families, parabolic implosion

Consider the following heuristic statement, concerning families of one-dimensional holomorphic dynamics: "Parabolic points are the source of instability." In this introduction, we will formulate a few theorems that motivate it.

Consider a family of rational maps $R_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ depending analytically on a parameter λ that belongs to a complex manifold Λ. Let the *stability set* Ω be the set of λ for which R_λ does not undergo a bifurcation, i.e. is conjugate on its Julia set to all nearby maps in the family. Define the bifurcation set B as the complement of Ω. The most famous example is $B = \partial M$, the boundary of the Mandelbrot set, for the family $R_\lambda(z) = z^2+\lambda$.

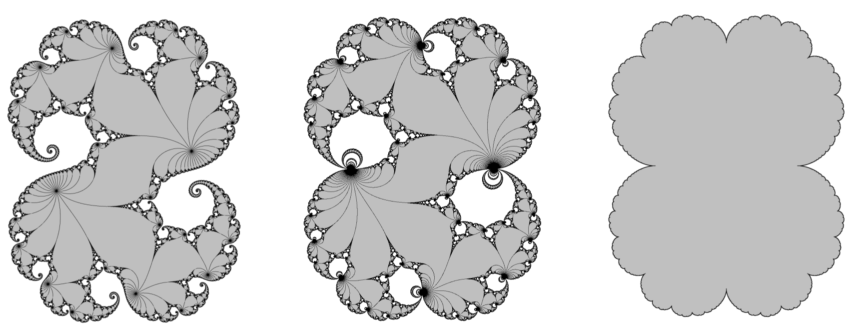
Theorem : (Mañé, Sad, Sullivan [32])

1. *Ω is open and dense,*
2. *B is the closure of the set of parameters λ for which the family has a non persistent parabolic point.*

A parabolic point is called *persistent* if nearby maps in the family still have a nearby periodic point with the same multiplier. We refer to [32] for more precise statements. In this report, we call *parabolic parameters* those for which R_λ has a non-persistent parabolic point λ.

If $B \neq \emptyset$, then the set B also contains many non-parabolic parameters. However, the parabolic parameters are those for which the discontinuity of the dynamical system is the worst. First, consider the following theorem, concerning the filled-in Julia sets $K(P_\lambda)$ for holomorphic families of polynomials P_λ:

Theorem : (Douady [17]) *The map $\lambda \mapsto K(P_\lambda)$ is upper semi-continuous for the Hausdorff distance on the set of compact subsets of $\widehat{\mathbb{C}}$, discontinuous at all non-persistent parabolic parameters, and continuous at all other parameters.*



The set of pictures above illustrates the discontinuity case of the theorem: let $P_\theta(z) = e^{2\pi i\theta}z + z^2$. On each image, black denotes the Julia set and gray denotes $K - J$. On the right we have $K(P_0)$, on the left, $K(P_{1/22})$ and in the middle, $\lim K(P_{1/n})$ together with $\lim J(P_{1/n})$. Note also the upper semi-continuity, the filled Julia set $K(P_0)$ of the limit contains $\lim J(P_{1/n})$.

For rational maps, there is no filled-in Julia set. A similar statement can be formulated and the discontinuity will occur iff there is a parabolic point, Siegel disk or Herman ring, (in each case we mean non-persistent). However, the discontinuity generated by parabolic points is much richer.

The first precise study of parabolic implosion was undertaken by Douady and Lavaurs in the case of quadratic and cubic polynomials. As an illustration of these results follows below a precise statement for the family of quadratic polynomials P_λ as above. Let $\lambda_0 = 1$ (similar statements follow for λ_0 a different root of unity, but these are slightly more involved). Let $\phi : B(0) \to \mathbb{C}$ denote an attracting Fatou-coordinate for P_{λ_0} where $B(0)$ denotes the parabolic basin of 0. Similarly let $\psi : \mathbb{C} \to \mathbb{C}$ denote a repelling Fatou parameter for P_{λ_0}. Finally for $A \in \mathbb{C}$ denote by

$$g_A := \psi(A + \phi(z)) : B(0) \to \mathbb{C}$$

the family of Lavaurs maps for P_{λ_0}.

Theorem : (Lavaurs [27]) *Given any $A \in \mathbb{C}$ there exists a sequence of parameters $\lambda_n \longrightarrow \lambda_0$ and there exists a sequence of corresponding integers $N_n \to +\infty$ such that the N_n-th iterate of P_{λ_n} tends, on the parabolic basin of P_{λ_0}, to the Lavaurs map g_A.*

Note that Lavaurs gave in fact an explicit sufficient condition on the sequences λ_n and N_n, depending on A, for the above convergence to occur: write $\lambda_n = e^{2\pi i \alpha_n}$ with α_n a complex number sequence tending to 0; then the condition is that

$$\frac{\pm 1}{\alpha_n} + N_n \longrightarrow A + a$$

where $a \in \mathbb{C}$ is another constant that depends in an explicit way on the choice of Fatou coordinates.

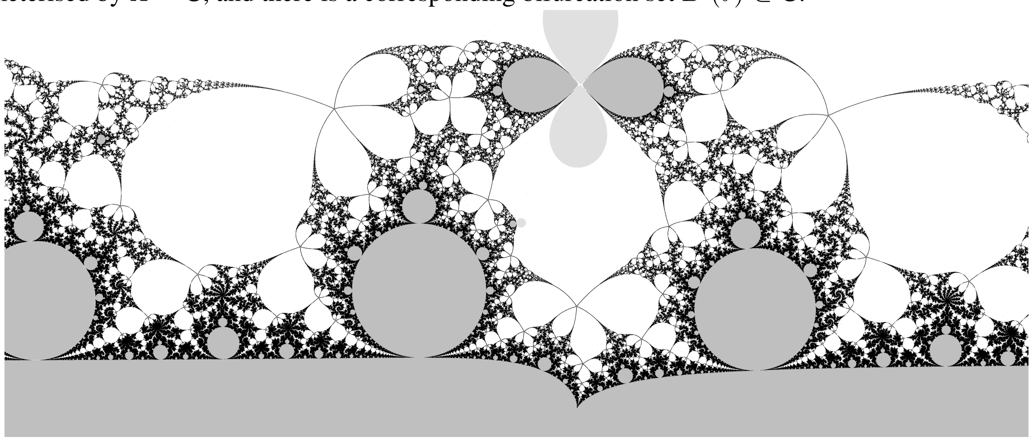
The convergence above of higher and higher iterates of nearby polynomials to Lavaurs's maps of the limiting polynomial is very similar to the phenomenon called *geometric limits* for Kleinian groups, see [43, 26]. For this reason Lavaurs maps are also called geometric limits. Presumably Lavaurs was aware that his results had generalizations to generic perturbations in families with a non persistent parabolic point, though he only gave a description of these limits in particular cases.

Here is a small list of striking consequences:

- (Douady [17], Lavaurs [27], late 1980's) The limit of the Julia set is strictly bigger than the Julia set of the limit. It corresponds to the Julia set of an enriched dynamical system, containing the limit map and the Lavaurs map. (This phenomenon is called *parabolic implosion*)
- (Lavaurs [27], late 1980's) Non-local connectivity of the cubic connectedness locus (analog of the Mandelbrot set for cubic polynomials)
- (Lavaurs [27]) Limit shape of some specific zooms on the Mandelbrot set.
- (Shishikura [39, 40], early 1990's) The Hausdorff-dimension of the boundary of the Mandelbrot set is equal to 2.
- (Douady, Hubbard, [20] mid 1980's) Discontinuity of renormalisation for polynomial-like maps.
- (Douady, Sentenac, [19] 1980's) Existence of external rays tending to the root of hyperbolic components of the Mandelbrot set.
- (Buff-Chéritat, [10], late 2000's) Fine estimates on the size of Siegel disks.
- (Buff-Chéritat, [9, 11], mid 2000's) Existence of Julia sets with positive Lebesgue measure.

Later work has focused on understanding more general bifurcations of parabolic points than the particular cases studied by Lavaurs and on developing the theory for generic families of holomorphic maps with a non-persistent parabolic orbit. This includes, but is not limited to work by Douady Estrada and Sentenac [18], pursued by Branner and Dias [7] (studying polynomial vector fields, a model and tool for a parabolic point), Oudkerk [34] (gate structures associated to a given perturbation), Christopher and Rousseau [38, 16] (classifying the full unfolding of the underlying bifurcation in general). Bedford-Smillie-Ueda [6] looked at parabolic implosion in a special case, *semi-parabolic* points, where the theory has a lot in common with the one-dimensional case, but proofs are harder: here one works in $\mathbb{C}^2$ and assumes that the eigenvalues of the fixed point are $\{1, \lambda\}$ with $|\lambda| < 1$. Few people have looked at the perturbation of maps tangent to the identity (all eigenvalues are 1).

Parabolic implosion also occurs in parameter space: one can consider the bifurcation set $B(\theta)$ for the family $P_\lambda(z) = e^{2\pi i\theta}z + \lambda z^2 + z^3$ ($\lambda \in \mathbb{C}$). These families are one dimensional slices in the 2-parameter family of all cubic polynomials. Note that if B denotes the bifurcation set for the latter, then $B(\theta)$ is contained in the θ-slice of B, but is not necessarily equal. Nonetheless, the difference can be precisely described. Similarly, one can look at quadratic rational maps with a fixed point of multiplier $e^{2\pi i\theta}$. It can also be parameterised by $\Lambda = \mathbb{C}$, and there is a corresponding bifurcation set $B'(\theta) \subset \mathbb{C}$.



In the picture above, black represents part of $B'(2/5)$. Gray indicates copies of the Mandelbrot set and light gray denotes parameters for which the parabolic point is parabolic-attracting (see [33]).

Describing $B(\theta)$ has attracted a lot of attention in recent work (Petersen-Roesch [37] for $B'(0)$, unpublished works by Roesh and Roesh-Nakane, Zakeri [49] for $B(\theta)$ where θ is a bounded type irrational). Eva Uhre [44] has studied similar slices, for quadratic rational maps. See also the work of Buff-Écalle-Epstein [12] and Petersen (double implosion, work in progress) concerning the parameters for which the number of petals doubles in $B'(1/n)$.

A striking feature is the following: If θ tends to p/q, the set $B(\theta)$ has a richer limit than just $B(p/q)$, and this limit is also described by parabolic implosion. This is the subject of study of I. Zidane (thesis, work in progress). Similar phenomena occur for $B'(\theta)$.

1.3 Parabolic renormalisation

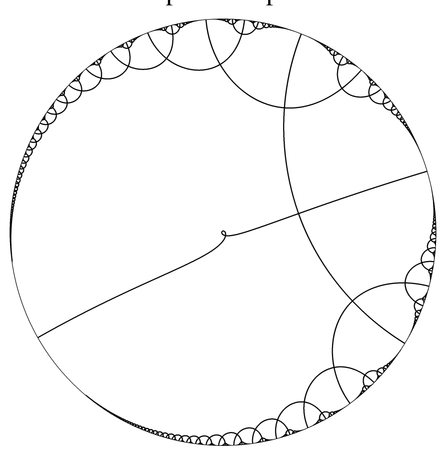
There may be further geometric limits occurring, depending on the fine behaviour of the perturbation. If this happens then the corresponding horn map has a Siegel or parabolic point. These second order geometric limits are defined on the basin of the parabolic point of the first order geometric limit, thus on a strict subset of the basin of the original parabolic point. The second order geometric limits (of parabolic type) played a crucial role in Shishikura's proof that the boundary of the Mandelbrot set has Hausdorff dimension 2, [39, 40]. There may also be third order geometric limits, and there exist even arbitrarily large possibly infinite order geometric limits, so-called towers. This led Adam Epstein [21] to develop in his thesis a formal framework and theory for formulating and proving statements not only about towers, but for general families of *finite type maps* and finitely generated dynamical systems from such maps. Following Epstein a finite type map is a co-compact holomorphic map with finitely many singular values (values which do not possess an evenly covered neighbourhood). This class of maps generalizes the class of rational maps. Finite type maps have rigidity properties that allow to extend many classical results. Epstein's theory has proven to be very fruitful: not only does it provide a precise frame work for discussing parabolic implosion, it also has produced strong results of its own such as the Fatou-Shishikura inequality for general finitely generated finite type dynamical systems, the proof of which specializes to a new proof of the Fatou-Shishikura inequality for rational maps, and generalizes the tools developed by Thurston in order to characterize post critically finite branched coverings. Also Epstein's theory has led to general transversality theorems.

A horn map of a finite type map has a natural maximal extension, which is again a finite type map. The extended horn maps have a dynamics that mimics (by a semi-conjugacy) the dynamics of the geometric limits. And there is a clear advantage in looking at the dynamics of the extended horn maps.

The set of all possible limits is an interesting object in itself. Epstein was developing his theory from the point of view of geometric limits, expressed in the language of sheaves. For an other perspective, Hubbard now considers the Julia sets as points in the space of compact sets and looks at the closure of the set of all Julia sets of $z^2 + c$, exploring its topological features. Bachy's thesis [5] explored one branch in this project: the Siegel disks (the other branch are the parabolic points).

Consider the operator that, to a parabolic point, associates say its (upper) horn map. We normalize Fatou coordinates so that the upper end of the fundamental cylinder is parabolic with multiplier 1 (the case of lower end is similar). Then the operator is called the (upper horn map) *parabolic renormalisation operator*. It can also be seen as the limit of a procedure of cylinder renormalisation (see the work of Yampolsky on circle maps [45, 46]; it is also a central heuristics in the work of Yoccoz on indifferent germs and circle maps [47], and the work of Perez-Marco on Hedgehogs, partially unpublished).

Shishikura [41] was the first to produce an invariant class SH_1 of maps for the parabolic renormalization operator for horn maps. Following this idea Petersen observed that there is a similar invariant class of "belt maps" for parabolic renormalization of critical circle maps with a parabolic fixed point. This observation was the starting point for Yampolsky's study of (parabolic) renormalization of critical circle maps [45]. Later Shishikura has defined a renormalization invariant subclass $SH_2 \subset SH1$, for which he jointly with H. Inou has developed a theory of near-parabolic renormalization. This theory in turn provided a missing link in the Douady-Cheritat strategy to prove existence of quadratic Julia sets with positive area [11]. Recently Cheritat has generalized the Inou-Shishikura theory of near-parabolic renormalization to families of maps with multiple critical point [15]. Yampolsky defined an invariant class for the (upper) horn map parabolic renormalization operator, which is slightly smaller than SH_1 and which allowed him to give an independent proof of the existence of a renormalization fixed point the parabolic renormalization operator.



The *Parabolic chessboard*, illustrated above, helps understanding the covering properties of maps in the invariant class. This proved useful in later works.

This striking feature hinted at contraction properties of the parabolic renormalisation, but actually showing this proved difficult. Inou and Shishikura [25] managed to do this by considering particular ramified coverings that contain only part of the structure. Their proof was computer assisted in 2002, they later gave a proof that can be checked by hand. Lanford III and Yampolsky developed a numerical scheme that enabled them to draw the domain of definition of the parabolic map f^*, the attracting fixed point of the renormalisation operator. Unfortunately Lanford-III passed away before the computer assisted proof was written up.

As a consequence of the contraction of parabolic renormalisation, and the techniques employed, it was possible to build a *near-parabolic renormalisation* invariant class for cylinder renormalisation [25], which proved particularly fertile in solving conjectures, especially for indifferent fixed points in the family of quadratic polynomials: control on the post-critical set (boundaries of Siegel disks, shape of hedgehogs, upper semi-continuity of closure of critical orbit is crucial in the proof of the positive measure theorem of Buff and Chéritat), size of Siegel disk and the Marmi-Moussa-Yoccoz conjecture, computability questions… People busy with this harvest include Buff, Cheraghi [14], Chéritat, Shishikura, Yampolsky, Zidane, with many collaborations between them.

It is to be noted that in [22], Yampolsky and Gaydashev gave a computer-assisted proof of the existence of a hyperbolic fixed point of cylinder renormalisation, with rotation number $\frac{\sqrt{5}+1}{2} = [1, 1, 1, \ldots]$ at one end of the cylinder, applicable to the family of quadratic polynomials. This particular example is outside the present reach of the techniques of Inou and Shishikura. Indeed the latter can only cover rotation numbers whose continued fraction entries are all bigger than an integer N whose value has not been determined (it should lie between 20 and 1000).

The Douady-Hubbard polynomial-like maps give rise to another notion of renormalization. Many polynomials P have iterates having polynomial-like restrictions f of degree equal or lower than the degree of P. Their Julia sets are attached to that of P at the pre-fixed points (points whose iterates are eventually fixed by f), which are either repelling or parabolic, and in the latter case *all attracting petals are in the filled-in Julia set of* f. Given a particular P, to prove the existence of f is usually obtained by cutting along external rays, equipotentials (see [19] for the definition of these notions) and modifying the construction near repelling fixed points. However, it has long been known that a similar construction should be possible when we do not want to keep all petals in the restriction. The problem is that f cannot be polynomial-like in this case. To handle this, Lomonaco developed in her thesis the notion of *parabolic-like* map and extended part of the Douady-Hubbard theory to this case, see [29, 30].

2 Current work and open problems

- **What are all limits?** As mentioned above, the set of all possible limits is under exploration by Epstein and Hubbard, with different approaches. To simplify, let us consider only the case of $P_c = z^2 + c$. Epstein looks at all possible limits of the set of iterates, i.e. all limits of sequences $P_{c_n}^{N_n}$, as $c_n \longrightarrow c$ and N_n is any sequence of integers. He develops a sheaf-theoretic flavored approach, that bears many significant subtleties. Hubbard wants to describe the closure, in the space of all compact subsets of the complex plane (or the Riemann sphere), endowed with the Hausdorff topology, of the set of all Julia sets $J(P_c)$, $c \in \mathbb{C}$.

  Let us mention that *rescaling limits* is the keyword for an even wider project: understanding all possible limits of (finite or infinite iterates)/(Julia sets), where one is allowed to rescale, i.e.: $a_n \circ P_{c_n}^{N_n} \circ a_n^{-1}$ or $a_n(J(f_n))$ where a_n is a complex-affine map of $\mathbb{C}$.

- **The general parabolic implosion.** There remains open questions in the classification of unfolding: Rousseau obtained a set of invariants that is complete in the sense that two unfoldings are equivalent iff their invariant are the same. However, the set of all values that can be taken by the invariant is not known. In other words there is a map $\mathcal{U} \to \mathcal{I}$ that is injective but its image still has to be determined.

  One also should look at the consequences of the above theory on the global dynamics of perturbation of parabolic points. In particular, it has consequences on the description of bifurcation loci in parameter spaces, near parabolic points.

- **Near parabolic renormalization.** The consequences of the Inou-Shishikura invariant class (that applies in particular to degree 2 polynomials) are still under study. In particular: for high type[1] rotation numbers θ, prove that the closure of the critical orbit of $P_\theta : z \mapsto e^{2\pi i\theta} z + z^2$ is a Jordan curve when the rotation number satisfies Herman's condition (see [48] for the definition of Hermans's condition) or a *Cantor Bouquet* otherwise (see for instance [1] for the definition of a Cantor Bouquet), determine its Hausdorff dimension, find a topological model for the whole Julia set, understand the ergodic properties for various measures (for instance the Lebesgue measure for those which have a positive one).

  Another promising direction is progresses on the famous MLC conjecture, that states that "the Mandelbrot set is Locally Connected". We are still very far from it but at least, the I.S.-near parabolic renormalization seems to be the right tool to attack the infinitely satellite case, at least partially. In this direction, see the work of Cheraghi and Shishikura (in preparation).

[1] This means that, for some integer N given by the theory, the entries of the continued fraction expansion $\theta = a_0 + 1/(a_1 + 1/\ldots)$ are required to satisfy $a_n \geq N$ for all $n \geq 1$.

Also, for θ having bounded type, Petersen [35] and McMullen [31] proved many properties of the Julia set (Local connectivity, density) and of the boundary of the Siegel disk (asymptotic self similarity). These proof rely heavily on the fact that there is a Quasi-conformal model for the dynamics (Ghys (Unpublished), Swiatek [42], Herman, (Unpublished), see also [36]). Chéritat and Yang Fei are investigating the possibility of reproving some of these results without the use of a quasiconformal model. The motivation is that for the family $e^z + c$, there cannot be such a model, but experimentally there seems to be an asymptotic self similarity.

In this spirit, Chéritat has recently tried to give a proof of a version of the Inou-Shishikura result, that would apply to $z^d + c$ (preprint, [15]). This work has not been peer reviewed. Hopes are that similar ideas could give something for $e^z + c$.

This is only the beginning of a broader story, as there are at least two directions for extending this approach:

- Build a useful near-parabolic renormalization operator for maps with 2 (or any $k > 1$) singular values, so that it applies to polynomials or rational maps of higher degree. The repelling direction of this operator would have $\mathbb{C}$-dimension 2 (resp. k) instead of 1 in the classical case. Part of Zidane's ongoing thesis is concerned with part of this program. See also the section below about fixed-point slices.
- Prove that the near parabolic operator (for $k = 1$ or bigger) extends to a sufficiently big class of maps to cover all rotation numbers, an not only the high type ones. This has also application to infinitely satellite renormalizable polynomials and the MLC conjecture.

- **Fractal analysis.** Maja Reisman and her collaborators are studying the box dimension of orbits in parabolic petals for the standard and also more refined gauges. This leads functional equations similar to the Fatou-Abel equation and to subtle and surprising estimates. This problem can find applications in the interpretation of pixel counting experiments.

- **Understanding resummation.** As mentioned earlier in this report, Sauzin, Lopez, Bouillot, Menous and many others are exploring the consequence of Écalle's work on parabolic points (and other systems), and are trying to make it more accessible. One task is to write down complete proofs (in some cases, Écalle only gave hints). Another is to understand what it tells us on dynamical systems. The Écalle coefficient depend holomorphically but not algebraically on the coefficients. One of the objective in Bouillot's work is to understand this dependence, or, loosely stated: how is it structured? In the earlier work [12], some characteristic of this dependence was used to get information on the repartition of a specific sequence of dynamically defined points in the parameter space of degree 2 rational maps.

  Conversely, what can the dynamics tell us on the Écalle coefficients?

- **Fixed-point slices.** Eva Uhre has been studying the slices $\mathrm{Per}_1(e^{2\pi ip/q})$ of the parameter space of quadratic rational maps. They are defined as the set of such maps that have a fixed point of multiplier $e^{2\pi ip/q}$. They form a one complex dimensional family, parameterizable by $\mathbb{C}$. Ideally, one would like to prove that the parameter picture is the mating between the Mandelbrot set minus a limb and the filled-in Julia set K of $z \mapsto e^{2\pi ip/q}z + z^2$ modified by some surgery. The part corresponding to the interior of the modified K is completely proved, see [44] which also includes some points in the boundary.

  In the case of cubic polynomials, the description of $\mathrm{Per}_1(e^{2\pi ip/q})$ (also parameterizable by $\mathbb{C}$) is different. P. Roesch has given a description (manuscript in preparation). Roesch and Nakane are currently working on the elucidation of the possible limits and accumulation sets of the Branner-Hubbard stretching rays (see [8]).

  For a parameter in $\mathrm{Per}_1(e^{2\pi ip/q})$ (in the cubic polynomials family or in the quadratic rational maps family), the parabolic point has either $2q$ or q petals because of a theorem of Fatou that states that every cycle of petals must contain a critical point in its basin. The set $D_{p/q}$ of parameters for which this number is $2q$ is finite. This set is of particular interest and has been studied by several people. For θ a bounded type irrational, the analog of this set in $\mathrm{Per}_1(e^{2\pi i\theta})$ is the set Z_θ of parameters for which both critical points are on the boundary of the Siegel disk. Zakeri [49] proved that Z_θ is a Jordan curve. The next steps would be:

- try to extend Zakeri's theorem to θ a Brjuno number (see [48] for the definition)
- try to figure out what could be the analog when θ is a non-Brjuno irrational: then there will be no Siegel disk at all for most values of the parameter
- understand the interactions between the sets Z_θ and $D_{p/q}$: for instance if p_n/q_n denotes the convergents of θ, does D_{p_n/q_n} tend to Z_θ for the Hausdorff topology on compact subsets of $\mathbb{C}$?
- There are naturally associated sub-harmonic functions whose laplacian are supported by $D_{p/q}$, resp. Z_θ. They may help in studying the questions above.

Several of these questions are studied in Zidane's thesis.

- **Enrichment in slices.**

  Let θ_n tends to a rational p/q in such a way that the corresponding cylinder renormalization has a rotation number that is independent of n. In other words, $p/q = [a_0; a_1, \ldots, a_k]$ and $\theta_n = [a_0; a_1, \ldots, a_k, n+\theta]$ for some $\theta \in \mathbb{R}$. Then one should be able to describe the limits of the parameter slice $\mathrm{Per}_1(e^{2\pi i\theta_n})$. Of particular interest are the limits of Z_{θ_n} or (D_{θ_n} if $\theta \in \mathbb{Q}$). For $\theta =$ the golden mean, computer experiments show an interesting phenomenon occurring. Zidane is studying this phenomenon. Buff and Écalle and Epstein studied in [12] the case $(\theta = 0, p/q = 0)$. Petersen is reinterpreting and extending their work by considering double parabolic implosion or equivalently the unfolding of simply degenerate parabolic points (work in progress).

- **Higher dimensions.** One of the major achievements of Sullivan in dynamics is the proof, that there are no wandering Fatou components for polynomials in one dimensional complex dynamics. The question was open in more dimensions until recently: Astorg, Buff, Dujardin, Peters and Raissy proved in [4] that wandering Fatou components occur for some polynomial endomorphism of $\mathbb{C}^2$. Their example is a fibered system (skew product) $(w, z) \mapsto (P(w), P_w(z))$ where P is polynomial, with a parabolic fixed point at the origin, and P_w is a perturbation of apolynomial P_0 that also has a parabolic fixed point at the origin. The question remains open in the case of polynomial automorphisms.

  Ueda, Smillie and Bedford are studying the extended Fatou coordinates and horn maps associated to a fixed point of a polynomial automorphism of $\mathbb{C}^2$ such that one eigenvalue has modulus < 1 and the other eigenvalue is equal to 1, see [6]. The study of the dynamic of nearby automorphisms (parabolic implosion) is also under way.

  The dynamics of fixed points whose linear part is the identity is much harder in dimension at least 2. We mentioned earlier the survey [2], and a few names. Good progress has been made, concerning the existence of petals (open or sub-manifolds) in specific direction but there are still a lot of dark points, and the full local dynamics still is unknown. The dynamics of perturbations (parabolic implosion) does not seem to have been studied much.

3 Outcome of the meeting

The meeting presented us with a selection of talks and mini-courses about hot topics and classical works, and allowed the present people to keep up to date with recent progress and established theory. In particular, it was nice to see that the field is still active, that a lot of open questions exists and that many of them seem addressable. The audience consisted in a mix of young and more experienced researchers, from all around the world. Almost all of the 19 participants gave a lecture: in alphabetic order we had Bedford, Bouillot, Chéritat, Epstein, Inou, Lomonaco, Mukherjee, Peter, Petersen, Reisman, Roesch, Rousseau, Shishikura, Uhre and Yampolsky. The list of talks and their abstract can be found elsewhere in this book. In the following we now list a few highlights among the outcomes of this meeting:

Hubbard and Epstein gave an interesting view on the ongoing endeavour consisting in trying to describe of all geometric limits of degree 2 polynomials (or degree d polynomials or rational maps).

Understanding Écalle's invariants are not less of a long task, and Bouillot's talk gave us a interesting glimpse involving beautiful, yet dense, formulas with multizetas and other fascinating quantities.

Inou and Mukherjee both studied non-local connectivity features of the *tricorn* (bifurcation/connectedness locus of the family $z \mapsto \overline{z}^2 + c$, generalization of which leads to an interesting open question: if two polynomials have parabolic points whose horn maps are equal, then what can one say about the relationship of the polynomials (for instance: are they necessarily semi-conjugates of a common polynomial?)

Yampolsky made a point that for renormalization problems there are often many different invariant subspaces to start from. He introduced two new such invariant space for circle maps. He particularly advocated the use of a specific invariant space for renormalization of commuting pairs. This space consists of holomorphic pairs, which commute up to third order Taylor coefficients. He showed how starting from this space one can obtain new and old renormalization results with much less effort than starting from the other invariant spaces which have been used over time.

A collaboration has begun between A. Chéritat and C. Rousseau about generic one-dimensional slices of unfoldings.

References

[1] J. M. Aarts, L. G. Oversteegen, The geometry of Julia sets, *Transactions of the American Mathematical Society*, **338** (1993), no. 2, 897–918.

[2] M. Abate, Fatou flowers and parabolic curves, *preprint, arXiv:1501.02176* (2015).

[3] M. Abate, Discrete holomorphic local dynamical systems. New version. In *Holomorphic dynamical systems*, Eds. G. Gentili, J. Guenot, G. Patrizio, Lect. Notes in Math. 1998, Springer, Berlin, 2010, pp. 1–55.

[4] M. Astorg, X. Buff, R. Dujardin, H. Peters, J. Raissy, A two-dimensional polynomial mapping with a wandering Fatou component *preprint, arXiv:1411.1188* (2014).

[5] I. Bachy, Enrichissements de Siegel, *thesis*, Univ. Aix-Marseille 1, (2011).

[6] E. Bedford, J. Smillie, T. Ueda, Parabolic Bifurcations in Complex Dimension 2, *preprint, arXiv:1208.2577* (2012).

[7] B. Branner, and K. Dias, Classification of complex polynomial vector fields in one complex variable, *Journal of Difference Equations and Applications* **16** (2010), no. 5-6, 463–517.

[8] B. Branner and J. H. Hubbard, The iteration of cubic polynomials. I. The global topology of parameter space, *Acta Mathematica* **160** (1988), no. 3–4, 143–206.

[9] X. Buff, A. Chéritat, Ensembles de Julia quadratiques de mesure de Lebesgue strictement positive, *Comptes Rendus Mathématique. Académie des Sciences. Paris*, **341** (2005), no. 11, 669–674.

[10] X. Buff, A. Cheritat, Sur l'implosion parabolique, la taille des disques de Siegel & une conjecture de Marmi, Moussa et Yoccoz (2008), Habiltation thesis of A. Chéritat, Université Paul Sabatier (Toulouse, France).

[11] X. Buff, A. Chéritat, Quadratic Julia sets with positive area, *Annals of Mathematics. Second Series* **176** (2012), no. 2, 673–746.

[12] X. Buff, J. Écalle and A. Epstein, Limits of degenerate parabolic quadratic rational maps, *Geometric and Functional Analysis* **23** (2013), no. 1, 42–95.

[13] C. Camacho, On the local structure of conformal mappings and holomorphic vector fields in $\mathbb{C}^2$. Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978), pp. 3, 83–94, Astérisque, 59-60, Soc. Math. France, Paris, 1978.

[14] D. Cheraghi, Typical orbits of quadratic polynomials with a neutral fixed point: Brjuno type, *Communications in Mathematical Physics* **322** (2013), no. 3, 999–1035.

[15] A. Chéritat, Near parabolic renormalization for unisingular holomorphic maps, *preprint, arXiv:1404.4735* (2014).

[16] C. Christopher and C. Rousseau, The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point, *International Mathematics Research Notices. IMRN* **9** (2014), 2494–2558,

[17] A. Douady, L'ensemble de Julia dépend-il continûment du polynôme?, In *Aspects des systèmes dynamiques*, 125–166, Ed. Éc. Polytech., Palaiseau, (2009).

[18] A. Douady, F. Estrada and P. Sentenac, Champs de vecteurs po lynomiaux sur ℂ, Unpublished manuscript.

[19] A. Douady and J. H. Hubbard, Étude dynamique des polynômes complexes. Partie II, *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]* **85**, With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac, Université de Paris-Sud, Département de Mathématiques, Orsay (1985).

[20] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, **18** (1985), no. 2, 287–343.

[21] A. Epstein, Towers of Finite Type Complex Analytic Maps, PhD Thesis, City University of New York (1993).

[22] D. Gaidashev, M. Yampolsky, Cylinder renormalisation of Siegel disks, *Experimental Mathematics* **16** (2007), no. 2, 215–226.

[23] O. E. Lanford III and M. Yampolsky, Fixed point of the parabolic renormalization operator, *Springer Briefs in Mathematics*, Springer, Cham, (2014).

[24] P. Fatou, Sur les équations fonctionnelles, I. *Bull. Soc. Math. France* **47** (1919), 161–271.

[25] H. Inou, M. Shishikura, Near Parabolic Renormalization, *preprint*.

[26] S. P. Kerckhoff, W. P. Thurston, Noncontinuity of the action of the modular group at Bers' boundary of Teichmüller space. *Invent. Math.* **100** (1990), no. 1, 25—47.

[27] P. Lavaurs, Systèmes dynamiques holomorphes : explosion de points périodiques paraboliques, *PhD Thesis*, Paris-Sud University, France (199).

[28] L. Leau, Étude sur les équations fonctionelles à une ou plusieurs variables. *Ann. Fac. Sci. Toulouse* **11** (1897), E1–E110.

[29] L. Lomonaco, Parabolic-like maps, *Erg. Theory and Dyn. Syst.* (first view, published on-line the 3rd of July 2014), 1–27.

[30] L. Lomonaco, Parameter space for families of Parabolic-like mappings, *Adv. in Math.* **261C** (2014), 200–219.

[31] C. T. McMullen, Self-similarity of Siegel disks and Hausdorff dimension of Julia sets, *Acta Mathematica* **180**, (1998), no. 2, 247–292.

[32] R. Mañé, P. Sad and D. Sullivan, On the dynamics of rational maps, *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, **16**, (1983), 193–217.

[33] J. Milnor, Dynamics in one complex variable, Third Edition, *Annals of Mathematics Studies* **160**, Princeton University Press, Princeton, NJ (2006)

[34] R. Oudkerk, Richard, The parabolic implosion: Lavaurs maps and strong convergenc for rational maps, In *Value distribution theory and complex dynamics (Hong Kong, 2000)*, Contemp. Math. **303**, 79–105, Amer. Math. Soc., Providence, RI, (2002).

[35] C. L. Petersen, Puzzles and Siegel disks, In *Progress in holomorphic dynamics*, Pitman Res. Notes Math. Ser. **387**, 50–85, Longman, Harlow, 1998.

[36] C. L. Petersen, The Herman-Światek theorems with applications, In *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, 211–225.

[37] C. L. Petersen and P. Roesch, Parabolic tools, *Journal of Difference Equations and Applications* **16** (2010), no. 5-6, 715–738.

[38] C. Rousseau, Analytic moduli for unfoldings of germs of generic analytic diffeomorphisms with a codimension k parabolic point, *Ergodic Theory and Dynamical Systems* **35** (2015), no. 1, 274–292.

[39] M. Shishikura, The boundary of the Mandelbrot set has Hausdorff dimension two, In *Complex analytic methods in dynamical systems (Rio de Janeiro, 1992)*, Astérisque **222**, no. 7, 389–405

[40] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, *Annals of Mathematics. Second Series* **147** (1998), no. 2, 225–267.

[41] M. Shishikura, Bifurcation of parabolic fixed points, In *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, 325–363, Cambridge Univ. Press, Cambridge, (2000).

[42] G. Swiatek, Rational rotation numbers for maps of the circle, *Communications in Mathematical Physics* **119** (1988), no. 1, 109–128.

[43] W. P. Thurston, The Geometry and Topology of Three-Manifolds, Lecture notes, Unpublished (1980).

[44] E. Uhre, A model for the parabolic slices $\mathrm{Per}_1(e^{2\pi i p/q})$ in moduli space of quadratic rational maps, *Comptes Rendus Mathématique. Académie des Sciences. Paris* **348** (2010), no. 23-24, 1327–1330.

[45] M. Yampolsky, Hyperbolicity of renormalisation of critical circle maps, *Publications Mathématiques. Institut de Hautes Études Scientifiques* **96** (2002), 1–41.

[46] M. Yampolsky, Renormalisation horseshoe for critical circle maps, *Communications in Mathematical Physics* **240** (2003), no. 1-2, 75–96.

[47] J.-C. Yoccoz, Petits diviseurs en dimension 1. *Astérisque* **231** (1995), 3—88.

[48] J.-C. Yoccoz, Analytic linearization of circle diffeomorphisms, In *Dynamical systems and small divisors (Cetraro, 1998)*, Lecture Notes in Math. **1784**, 125–173, Springer, Berlin, 2002.

[49] S. Zakeri, Dynamics of cubic Siegel polynomials, *Communications in Mathematical Physics* **206** (1999), no. 1, 185–233.