Homogeneous ultrametric spaces

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joint work with

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Let $R := [0, +\infty[$ and $R_* :=]0, +\infty[$.

**Definition : Ultrametric space**

An *ultrametric* space is a pair $\mathbf{M} = (X, d)$ with $d : X \times X \to R$ s.t. :
1. $d(x, y) = 0 \iff x = y$ (separation).
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) = \max\{d(x, y), d(y, z)\}$ (strong triangular inequality).

A metric space is ultrametric if and only if its triangles are isosceles acute, if and only if any two meeting balls are comparable for inclusion.

**Examples**

1. $(R, \max)$, given by $d(x, y) = \max\{x, y\}$ if $x \neq y$ and $d(x, x) = 0$.
2. Baire and Cantor spaces, $\mathbb{Q}_p$, etc.
Ultrametric spaces

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#### Ultrametric space: balls

- **Balls of radius** $r \in \mathbb{R}$ **centered at** $x \in X$
  - $B_{<}(x, r) := \{ y \in X : d(x, y) < r \}$: open ball
  - $B_{\leq}(x, r) := \{ y \in X : d(x, y) \leq r \}$: closed ball

#### Basic properties

- Open balls are clopen, as well as closed balls of non-zero radius. Ultrametric spaces are totally-discontinuous.
- Meeting balls are comparable w.r.t. inclusion.
- Each point of a ball is a center: $y \in B(x, r) \Rightarrow B(x, r) = B(y, r)$.
- The diameter of a ball is the least of its radii. In particular: $\text{diam}(B_{\leq}(x, d(x, y))) = d(x, y)$. 
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The $R$-labeled tree $\text{Nerve}(M)$

Nerve of an ultrametric space: definition

A ball attains its diameter if and only if it is of the form $B_\leq(x, d(x, y))$. The diameter of such a ball is the radius:

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**Definition**

The *nerve* of $M$ is the collection of closed balls attaining their diameter:

$$\text{Nerve}(M) := \{B_\leq(x, d(x, y)) : x, y \in X\}$$

This is a *leafy tree* for the relation of inclusion:

- This is a join-semi-lattice, *i.e.*, every pair of nodes has a supremum:
  $$B_\leq(x, r) \lor B_\leq(x', r') = B_\leq(x, \max\{r, r', d(x, x')\})$$
- Any two nodes greater than a third one are comparable.
- Every node is a leaf or the join of two leaves:
  $$B_\leq(x, d(x, y)) = B_\leq(x, 0) \lor B_\leq(y, 0) = \{x\} \lor \{y\}$$

The distance of $M$ is recoverable from the nerve equipped with its diameter function $\text{diam} : \text{Nerve}(M) \to R$:

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Given an inner node of the nerve, $B$ of diameter $r > 0$,

$$y \sim z :\iff d(y, z) < r$$

is an equivalence relation on $B$. Its classes are the open balls of radius $r$ centered in $B$.

**Sons of an inner node of the nerve**

$$\forall x \neq y : \text{Sons } B_{\leq} (x, d(x, y)) := \{ B_{\leq} (z, d(x, y)) : z \in B_{\leq} (x, d(x, y)) \}$$

The degree of a member of the nerve is the number of its sons.

A son of a node of $\text{Nerve}(M)$ may fail to belong to $\text{Nerve}(M)$. It is indeed its son in the tree of all non-empty balls.

In $(R, \text{max})$, $\text{Sons}([0, 1]) = \{ [0, 1[, \{ 1 \} \}$. 
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The *degree* of a member of the nerve is the number of its sons.

A son of a node of $\text{Nerve}(\mathbf{M})$ may fail to belong to $\text{Nerve}(\mathbf{M})$. It is indeed its son in the tree of all non-empty balls.

In $(\mathbb{R}, \max)$, $\text{Sons}([0, 1]) = \{[0, 1[, \{1\}\}$. 
Nerve of an ultrametric space

Nerve of an ultrametric space: examples

Figure: A triangle and its nerve

Figure: The nerve of \((R, \text{max})\)
Nerve of an ultrametric space

Nerve of an ultrametric space: examples

\begin{figure}
\centering
\begin{tikzpicture}
  \node (a) at (0,0) [circle, fill=black] {$a$};
  \node (b) at (1,0) [circle, fill=black] {$b$};
  \node (c) at (0.5,1) [circle, fill=black] {$c$};
  \draw (a) -- (b) node [midway, left] {$\ell$};
  \draw (a) -- (c) node [midway, below] {$L$};
  \draw (b) -- (c) node [midway, above] {$L$};
\end{tikzpicture}
\caption{A triangle and its nerve}
\end{figure}

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  \node (c) at (0.5,1) [circle, fill=black] {$c$};
  \node (abc) at (0.5,2) [circle, fill=gray] {$\{a, b, c\}$};
  \draw (a) -- (b) node [midway, left] {$\ell$};
  \draw (a) -- (abc) node [midway, below] {$\{a, b\}$};
  \draw (b) -- (abc) node [midway, above] {$\{a, b\}$};
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\caption{The nerve of \((R, \max)\)}
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Nerve of an ultrametric space

Figure: The nerve of a finite ultrametric space
Ultrametric spaces

Ultrametric space
Examples: spaces of sequences and spaces of functions

Figure: The nerve of $\mathbb{Q}_2$ has no root
An ultrametric space $\mathbf{M}$ is \textit{spherically-complete} if every chain of non-empty balls has a non-empty intersection, if and only if every chain of the nerve has a lower bound.

Requiring above that the infimum of the diameters of the members of the chain be 0 yields Cauchy-completeness.

\textbf{Example}

$([1, \infty[. \max)$ is Cauchy- but not spherically-complete: 

$\bigcap_{x>1} B_{\leq} (x, x) = \emptyset.$

$\bigcap_{1,x}[\bigl] \not= \emptyset.$
Ultrametric spaces

Ultrametric space
Cauchy- and spherical completeness

Spherical completeness

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Motivation: indivisibility of $\mathbb{U}_Q \cap [0,1]$?

Definition (Indivisible metric space)

A metric space $\mathbf{M}$ is *indivisible* if it embeds in some class of each of its finite partitions: $(\mathbf{M} \rightarrow [\mathbf{M}]_k)$.

Question (Hjorth)

Is the bounded rational Urysohn space indivisible?

Theorem (DLPS07: no)

*Every Cantor connected metric space is divisible.*

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Two points $x$ and $y$ of a metric space $M$ are $\varepsilon$-chainable ($\varepsilon > 0$) if there is finite sequence of points ($x = z_0, z_1, \ldots, z_n = y$) such that $d(x_i, x_{i+1}) \leq \varepsilon$ ($0 \leq i < n$).

They are chainable if they are $\varepsilon$-chainable for every $\varepsilon$.

$M$ is Cantor connected if any two points are chainable.

$M$ is totally Cantor disconnected if any two points are unchainable.

Example: ultrametric spaces are totally Cantor disconnected.

Proposition (Lemin)

A metric space $M$ is totally Cantor disconnected if and only if its distance is $\geq$ a ultrametric one. In this case there is a greatest such ultrametric distance: $d_{\text{Ult}}(x, y)$ is the supremum of the witnessing $\varepsilon$'s.

Proposition

If $(X, d)$ is countable, homogenous and indivisible, then so is $(X, d_{\text{Ult}})$. 


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### Proposition

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Homogeneous ultrametric spaces

- A *local isometry* of a metric space \((X, d)\) is an isometry form a finite subspace of \(X\) into \(X\).

- The metric space is *(ultra)-homogeneous* if every local isometry extends to a bijective isometry of \(X\).

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*An ultrametric space is homogeneous as soon as it is transitive.*

The countable case is in [DLPS07], and the Polish case is in [Malicki-Solecki09].
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Proposition (Three equivalent formulations)

Consider an ultrametric space \( M = (X, d) \).

1. A local isometry extends to a bijective isometry of \( M \), as soon as each of its restrictions to singletons does.

2. \((X, X^2 \xrightarrow{d} R, X \to X/\text{Aut}(M))\) is homogeneous.

3. A mapping from a finite subset of \( X \) into \( X \) extends to an isometric bijection of \( X \), as soon as each of its restrictions to pairs does. (The canonical action of the isometry group of \( M \) has arity 2.)

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\psi : \begin{pmatrix}
S & S'' & \text{elsewhere} \\
\psi_c & \psi_A \psi_c^{-1} \psi_A & \psi_A
\end{pmatrix}
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The homogeneity of \((M, X \to X/\text{Aut}(M))\) holds more generally for every hereditarily decomposable symmetric binary relational structure \(M\).

A *module* of an irreflexive binary relational structure \((X, R_i : i \in I)\) is a set \(M\) of vertices of which the elements all look alike for each vertex outside.

The empty set, the singletons, and the vertex set are trivially modules.

A structure is *prime* if it has at least three vertices and all its modules are trivial.

A structure is *hereditarily decomposable* if it embeds no prime structure.

- Notice that a linearly ordered set is hereditarily decomposable but not symmetric.
- The hereditarily decomposable simple graphs are those that embed no path on four vertices.
- The Random graph is prime.

Each binary relational structure has a modular decomposition tree from which it is recoverable. For an ultrametric space, this is its nerve.
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The homogeneity of \((M, X \to X / \text{Aut}(M))\) holds more generally for every hereditarily decomposable symmetric binary relational structure \(M\).

A module of an irreflexive binary relational structure \((X, R_i : i \in I)\) is a set \(M\) of vertices of which the elements all look alike for each vertex outside.

The empty set, the singletons, and the vertex set are trivially modules.

A structure is prime if it has at least three vertices and all its modules are trivial.

A structure is hereditarily decomposable if it embeds no prime structure.

- Notice that a linearly ordered set is hereditarily decomposable but not symmetric.
- The hereditarily decomposable simple graphs are those that embed no path on four vertices.
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Figure: Besides the balls, the modules of an ultrametric space look like this.
Spectrum and degree sequence of an ultrametric space \( M = (X, d) \)

### Spectrum

The *spectrum* of a point \( x \in X \) is the set of the distances that it realizes:

\[
\text{spec}(x) := \{d(x, y) : y \in X\} \subseteq \mathbb{R}
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The *spectrum* of \( M \) is the set of all realized distances:

\[
\text{spec}(M) := \{d(x, y) : x, y \in X\} = \bigcup\{\text{Spec}(x) : x \in X\}
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### Degree sequence

\[
\text{Dgr} : \begin{cases} 
\text{spec}(M) \to \text{CARD}_* \\
r \mapsto \sup\{\text{card}(\text{Sons}(B)) : B \in \text{Nerve}(M), \text{diam}(B) = r\}
\end{cases}
\]

maps each element \( r \) of the spectrum to the supremum of the degrees of nodes of \( \text{Nerve}(M) \) of diameter \( r \).
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If an ultrametric space $M = (X, d)$ is homogeneous, then:

- its points have the same spectrum and
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Ultrametric spaces

**Ultrametric space** $M^I_\nu$

**Definitions**

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- a (degree) function $\nu : V_* \to \text{CARD}_* \subset \text{ON}_*$.

For $x$ and $y$ in $\text{ON}^{V_*}$, let:

- $\Delta(x, y) := \{r \in V_* : x(r) \neq y(r)\}$,
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**Proposition**

Letting $X^I_\nu := \{x \in \text{ON}^{V_*} : x < \nu$ and $\text{supp}(x) \in \mathcal{I}\}$:

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- $\text{spec}(M^I_\nu) = \mathcal{I}^\vee := \{\sup W : W \in \mathcal{I}\}$.
- If $\text{Fin}(V_*) \subseteq \mathcal{I} \subseteq \text{coWell}(V_*)$, then $Dgr(M^I_\nu) = \nu$.
  
- $M^I_\nu^{\text{Chy}} := M^{\ell \mathcal{I}}_\nu : \ell \mathcal{I} := \{W \subseteq V_* | \forall m > 0 : W \cap [m, +\infty[ \in \mathcal{I}\}$. 


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- $\overline{M^\mathcal{I}_\nu}^{\text{Chy}} = M^\ell_{\mathcal{I}} : \ell\mathcal{I} := \{W \subseteq V_* \mid \forall m > 0 : W \cap [m, +\infty[ \in \mathcal{I}\}.$
Ultrametric spaces

**Ultrametric space** $M^\mathcal{I}_\nu$

**Definitions**

Consider:

- a subset $V$ of $R$ containing 0 and let $V_* := V \setminus \{0\}$,
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**Proposition**

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**Ultrametric spaces**

**Ultrametric space** \( \mathbf{M}^I_\nu \)

**Definitions**

Consider:
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Ultrametric spaces 

**Ultrametric space** \( M^\mathcal{I}_\nu \)

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Ultrametric spaces

**Ultrametric space** $\mathbf{M}^\mathcal{I}_\nu$

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Ultrametric spaces $M^\mathcal{I}_\nu$

Definitions

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Ultrametric spaces

**Definitions**
Ultrametric spaces

Ultrametric space \( \mathbf{M}^I_{\nu} \)

Definitions

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Proposition

Letting \( X^I_{\nu} := \{ x \in \text{ON}^{V_*} : x < \nu \text{ and } \text{supp}(x) \in \mathcal{I} \} \):

- \( \mathbf{M}^I_{\nu} := (X^I_{\nu}, \text{sup} \Delta) \) is an ultrametric space;
  it is homogeneous and spherically complete.
- \( \text{spec}(\mathbf{M}^I_{\nu}) = \mathcal{I}^\nu := \{ \text{sup } W : W \in \mathcal{I} \} \).
- If \( \text{Fin}(V_*) \subseteq \mathcal{I} \subseteq \text{coWell}(V_*) \), then \( \text{Dgr}(\mathbf{M}^I_{\nu}) = \nu \).
- \( \mathbf{M}^{\text{Chy}}_{\nu} = \mathbf{M}^{\ell\mathcal{I}}_{\nu} : \ell\mathcal{I} := \{ W \subseteq V_* \mid \forall m > 0 : W \cap [m, +\infty[ \in \mathcal{I} \} \).
Consider:

- a subset $V$ of $R$ containing 0 and let $V_* := V \setminus \{0\}$,
- an ideal $\mathcal{I}$ of subsets of $V_*$ bounded above in $R$,
  - $\text{Fin}(V_*)$ : ideal of finite subsets of $V_*$
  - $\text{coWell}(V_*)$ : ideal of co-well founded subsets $W$ of $V_*$,
- a (degree) function $\nu : V_* \to \text{CARD}_* \subset \text{ON}_*$.

For $x$ and $y$ in $\text{ON}^{V_*}$, let:

- $\Delta(x, y) := \{r \in V_* : x(r) \neq y(r)\}$,
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**Proposition**

Letting $X^\mathcal{I}_\nu := \{x \in \text{ON}^{V_*} : x < \nu \text{ and supp}(x) \in \mathcal{I}\}$:

- $M^\mathcal{I}_\nu := (X^\mathcal{I}_\nu, \sup \Delta)$ is a an ultrametric space ;
  it is homogeneous and spherically complete.
- $\text{spec}(M^\mathcal{I}_\nu) = \mathcal{I}^\vee := \{\sup W : W \in \mathcal{I}\}$.
- If $\text{Fin}(V_*) \subseteq \mathcal{I} \subseteq \text{coWell}(V_*)$, then $\text{Dgr}(M^\mathcal{I}_\nu) = \nu$.
- $M^\mathcal{I}_\nu^{\text{Chy}} = M^{\mathcal{I}^{\circ \mathcal{I}}}_\nu : \mathcal{I}^{\circ \mathcal{I}} := \{W \subseteq V_* \mid \forall m > 0 : W \cap [m, +\infty[ \in \mathcal{I}\}$. 
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Embedding $M$ into $M_{\nu}^{\text{coWell}(\text{spec}(M)^*)}$

Properties

**Lemma**

*Each ultrametric space* $M = (X, d)$ *embeds in* $M_{\nu}^{\text{coWell}(V^*)}$ *for* $\nu = \text{Dgr}(M)$ *and* $V = \text{spec}(M)$.*

**Proof.**

Consider a well-ordering $<$ of $X$. For each inner node $B$ of $\text{Spec}(M)$, define the well-ordering $<_B$ of $\text{Sons}(B)$:

$$S_1 <_B S_2 :\iff \min < S_1 < \min < S_2.$$  

Then let $\rho_B$ map each son $S$ of $B$ to its rank

$$\rho_B(S) < \text{deg}(B) \leq \text{Dgr}_M(B)(\text{diam}(B))$$

for this ordering. Then map each $x \in X$ to

$$s(x) := (\rho_B(\text{son}(B, x)) : B := B_<(x, r), r \in \text{spec}(x)).$$

Observe that $B \in \text{supp}(s(x)) \mapsto \min < (\text{son}(B, x)) \in X$ is decreasing for $\supset$ and $<$. 
Embedding $\mathbf{M}$ into $\mathbf{M}_{\text{coWell}}^\nu(\text{spec}(\mathbf{M})_\ast)$

**Properties**

**Lemma**

Each ultrametric space $\mathbf{M} = (X, d)$ embeds in $\mathbf{M}_\nu^\text{coWell}(V_\ast)$ for

$\nu = \text{Dgr}(\mathbf{M})$ and $V = \text{spec}(\mathbf{M})$.

**Proof.**

Consider a well-ordering $<$ of $X$. For each inner node $B$ of Spec($\mathbf{M}$),
define the well-ordering $<_B$ of Sons($B$):

$S_1 <_B S_2 :\iff \min_<_S_1 < \min_<_S_2$.

Then let $\rho_B$ map each son $S$ of $B$ to its rank

$\rho_B(S) < \text{deg}(B) \leq \text{Dgr}_\mathbf{M}(B)(\text{diam}(B))$ for this ordering.

Then map each $x \in X$ to

$s(x) := (\rho_B(\text{son}(B, x)) : B := B_\preceq(x, r), r \in \text{spec}(x))$.

Observe that $B \in \text{supp}(s(x)) \iff \min_<(\text{son}(B, x)) \in X$ is decreasing for $

\supset$ and $<$.
Embedding $\mathbf{M}$ into $\mathbf{M}^{\text{coWell}}_{\nu}(\text{spec}(\mathbf{M})_*)$ for $\nu = \text{Dgr}(\mathbf{M})$ and $V = \text{spec}(\mathbf{M})$.

**Lemma**

Each ultrametric space $\mathbf{M} = (X, d)$ embeds in $\mathbf{M}^{\text{coWell}}_{\nu}(V_*)$ for $\nu = \text{Dgr}(\mathbf{M})$ and $V = \text{spec}(\mathbf{M})$.

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Consider a well-ordering $<$ of $X$. For each inner node $B$ of $\text{Spec}(\mathbf{M})$, define the well-ordering $<_B$ of $\text{Sons}(B)$:

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Then map each $x \in X$ to

$$s(x) := (\rho_B(\text{son}(B, x)) : B := B_{\leq}(x, r), r \in \text{spec}(x)).$$

Observe that $B \in \text{supp}(s(x)) \mapsto \min < (\text{son}(B, x)) \in X$ is decreasing for $\supset$ and $<$.
Embedding $\mathbf{M}$ into $\mathbf{M}_{Dgr(M)}^{coWell(spec(M)_*)}$

**Lemma**

Each ultrametric space $\mathbf{M} = (X, d)$ embeds in $\mathbf{M}_\nu^{coWell(V_*)}$ for $\nu = Dgr(M)$ and $V = spec(M)$.

**Proof.**

Consider a well-ordering $<$ of $X$. For each inner node $B$ of $\text{Spec}(\mathbf{M})$, define the well-ordering $<_B$ of $\text{Sons}(B)$:

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Then let $\rho_B$ map each son $S$ of $B$ to its rank $\rho_B(S) < \deg(B) \leq Dgr_{\mathbf{M}}(B)(\text{diam}(B))$ for this ordering.

Then map each $x \in X$ to $s(x) := (\rho_B(\text{son}(B, x)) : B := B_{\leq}(x, r), r \in spec(x))$.

Observe that $B \in \text{supp}(s(x)) \mapsto \min < (\text{son}(B, x)) \in X$ is decreasing for $\supset$ and $<$.
Embedding $M$ into $M^\text{coWell}(\text{spec}(M)_*)^{\text{Dgr}(M)}$

Properties

**Lemma**

*Each ultrametric space $M = (X, d)$ embeds in $M_\nu^\text{coWell}(V_*)$ for $\nu = \text{Dgr}(M)$ and $V = \text{spec}(M)$.*

**Proof.**

Consider a well-ordering $<$ of $X$. For each inner node $B$ of Spec$(M)$, define the well-ordering $<_B$ of Sons$(B)$:

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Then map each $x \in X$ to

$$s(x) := (\rho_B(\text{son}(B, x)) : B := B \leq (x, r), r \in \text{spec}(x)).$$

Observe that $B \in \text{supp}(s(x)) \mapsto \min_<(\text{son}(B, x)) \in X$ is decreasing for $\supset$ and $<$.
Lemma

Each ultrametric space \( M = (X, d) \) embeds in \( M^{\text{coWell}(V_* \uparrow \nu)} \) for \( \nu = \text{Dgr}(M) \) and \( V = \text{spec}(M) \).

Proof.

Consider a well-ordering \( \prec \) of \( X \). For each inner node \( B \) of \( \text{Spec}(M) \), define the well-ordering \( \prec_B \) of \( \text{Sons}(B) \) :

\[
S_1 \prec_B S_2 :\iff \min \prec S_1 < \min \prec S_2.
\]

Then let \( \rho_B \) map each son \( S \) of \( B \) to its rank

\[
\rho_B(S) < \text{deg}(B) \leq \text{Dgr}_M(B)(\text{diam}(B))
\]

for this ordering.

Then map each \( x \in X \) to

\[
s(x) := (\rho_B(\text{son}(B, x)) : B := B \leq (x, r), r \in \text{spec}(x)).
\]

Observe that \( B \in \text{supp}(s(x)) \mapsto \min \prec (\text{son}(B, x)) \in X \) is decreasing for \( \supset \) and \( \prec \).
Embedding $M$ into $M_{\text{coWell}}(\text{spec}(M)_*)$  

Properties

Lemma

Each ultrametric space $M = (X, d)$ embeds in $M_{\nu}^{\text{coWell}(V_*)}$ for $
u = \text{Dgr}(M)$ and $V = \text{spec}(M)$.

Proof.

Consider a well-ordering $<$ of $X$. For each inner node $B$ of $\text{Spec}(M)$, define the well-ordering $<_B$ of $\text{Sons}(B)$:

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Lemma

Each ultrametric space \( M = (X, d) \) embeds in \( M_{\nu}^{\text{coWell}(V_*)} \) for \( \nu = \text{Dgr}(M) \) and \( V = \text{spec}(M) \).

Proof.

Consider a well-ordering \( < \) of \( X \). For each inner node \( B \) of \( \text{Spec}(M) \), define the well-ordering \( <_B \) of \( \text{Sons}(B) \):

\[
S_1 <_B S_2 : \iff \min_< S_1 < \min_< S_2.
\]

Then let \( \rho_B \) map each son \( S \) of \( B \) to its rank

\[
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\[
s(x) := (\rho_B(\text{son}(B, x)) : B := B_<(x, r), r \in \text{spec}(x)).
\]

Observe that \( B \in \text{supp}(s(x)) \iff \min_(\text{son}(B, x)) \in X \) is decreasing for \( \supset \) and \( < \).
Consequences

- $M$ is homogeneous and countable (resp. and Polish) if and only if it is isomorphic to $M_{\nu}^{\text{Fin}(V_*)}$ (resp. to $M_{\nu}^{\ell\text{Fin}(V_*)}$) for some countable $V$ and $\nu \leq \omega$.

- (Feinberg) A spherically complete ultrametric space is homogeneous if and only if it is sub-homogeneous.

- $M$ is spherically complete and homogeneous if and only if it is isomorphic to some $M_{\nu}^{\text{coWell}(V_*)}$.

- (Delon) $M_{\nu}^{\text{Fin}(V_*)}$ embeds in every sub-homogeneous ultrametric space $M$ with $V \subseteq \text{spec}(M)$ and $\nu \leq \text{Dgr}(M)$.

- (Lemin) If $M$ has density at most $\kappa$ and spectrum included in $V$, then it embeds in $M_{\kappa}^{\text{coWell}(V_*)}$.

For each ordinal $\xi$, let $\text{coWell}(V_*, \xi)$ denote the ideal of those $W$ of type less than $\omega^\xi$. If $V \supseteq \mathbb{Q}^+$ and $\nu \geq 2$, then the family $(M_{\nu}^{\text{coWell}(V_*, \xi)} : \xi < \omega_1)$ is strictly increasing w.r.t. embeddability.
Consequences

- **M** is homogeneous and countable (resp. and Polish) if and only if it is isomorphic to \( M^\text{Fin}(V_\ast) \) (resp. to \( M^\ell\text{Fin}(V_\ast) \)) for some countable \( V \) and \( \nu \leq \omega \).

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**Definition (Local spec-isometry)**

A *local spec-isometry* of $\mathbf{M}$ is a local automorphism of the enriched structure $(X, X^2 \xrightarrow{d} R, X \xrightarrow{\text{spec}} \wp(R))$.

Thus a local spec-isometry is a local isometry that maps each point to a point with the same spectrum.

**Definition (Spec-homogeneity)**

The ultrametric space $\mathbf{M} = (X, d)$ is *spec-homogeneous* if every local spec-isometry of $\mathbf{M}$ extends to a bijective isometry.

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### Definition (Similar balls)

Two balls of the same radius and type ("open" or "closed") $B_1$ and $B_2$ are similar if they have points with the same spectrum:

$$\exists x_1 \in B_1, x_2 \in B_2 : \text{spec}(x_1) = \text{spec}(x_2).$$

### Theorem

An ultrametric space is spectral-homogenous if and only if any two similar balls are isomorphic.

### Corollary

An ultrametric space is homogenous if and only if all points have the same spectrum and any two balls of the same type ("open" or "closed") and the same radius are isomorphic.

### Corollary

The Cauchy-completion of a homogeneous ultrametric space is homogeneous.
Spectral homogeneity
Characterization

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Proposition

A countable ultrametric space is spectral-homogenous if and only if any two similar members of the nerve are isomorphic.

Corollary

In particular a countable ultrametric space is spectral-homogenous if any two members of the nerve with the same diameter are isomorphic.

Question

Can the countability condition in the above statements be lifted?

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A countable ultrametric space extends to a spec-homogeneous countable ultrametric space.
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Therefore $(C_1, C_2) \mapsto \text{dist}(C_1, C_2)$ is an ultrametric distance on $X/\text{Aut}(M)$.

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\[ 0 \leftarrow \cdots < d(\infty, x_n) = r_n < \cdots < d(\infty, x_0) = r_0 \]

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\phi_n : \begin{cases} 
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  \phi_n \upharpoonright X \setminus B_{\leq} (x_n, r_n) = \text{id} 
\end{cases}
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