Indivisible groups

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Outline

Let R be a structure on a set U.

A copy of R is an induced substructure C of R which is isomorphic to R.

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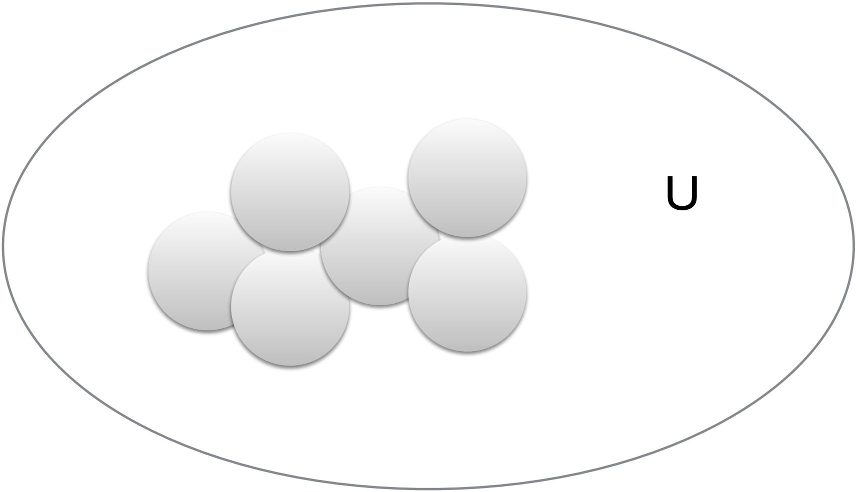
The isomorphism of R to C is an *embedding* of R.

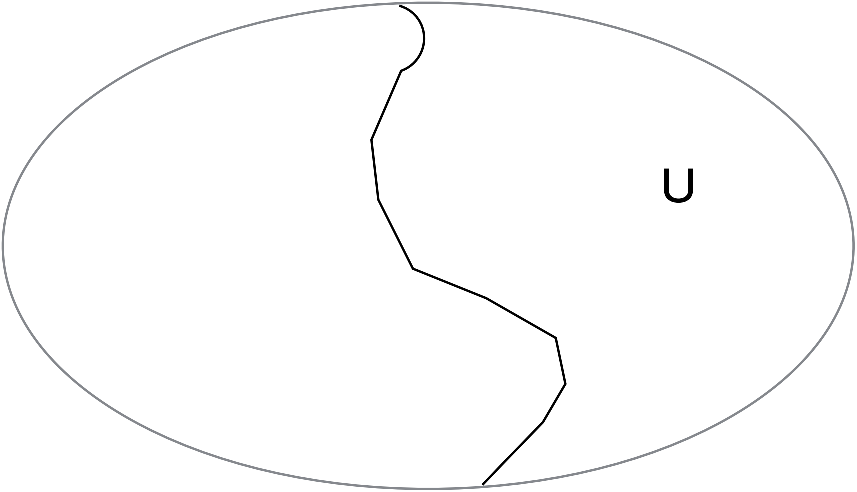

U

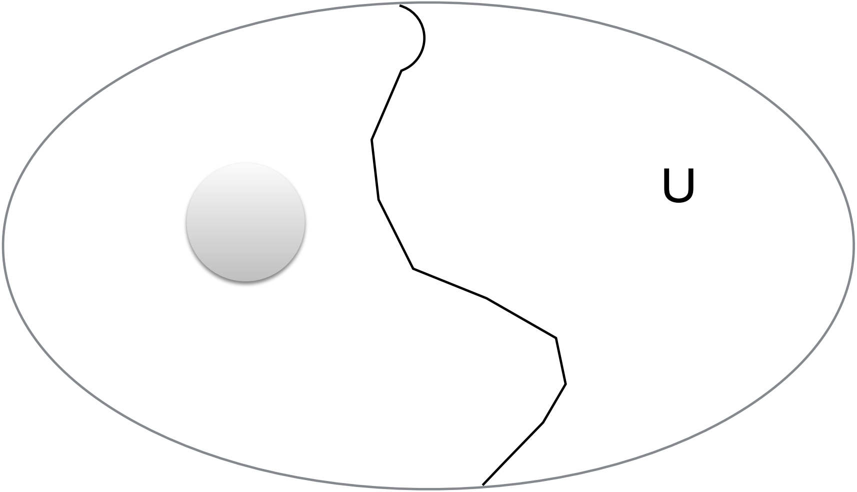
A structure R on U is *indivisbile* if:

For every partition (P_0, P_1) of U

there exists a copy in P_0 or in P_1.

The structure is *divisible* if it is not indivisible.


U


U

U will denote a countable infinite set and

G a subgroup of the symmetric group of U.

The group G acts transitivley on U.

An injection f of U into U is an *embedding* of $\mathbb{G}$ if:

For every finite $A \subseteq U$ exists a function $g \in \mathbb{G}$ for which

the restriction of g to A is equal to the restriction of f to A.

An injection f of U into U is an *embedding* of G if:

For every finite $A \subseteq U$ exists a function $g \in \mathrm{G}$ for which the restriction of g to A is equal to the restriction of f to A.

The image of an embedding of G is a *copy* of G.

The group G is *indivisible* if:

For every partition of U into two parts

there exists a copy in one of the parts.

The group G has *arity* less than or equal to $n \in \omega$ if

for all finite subsets A and B of U and bijections $f : A \to B$:

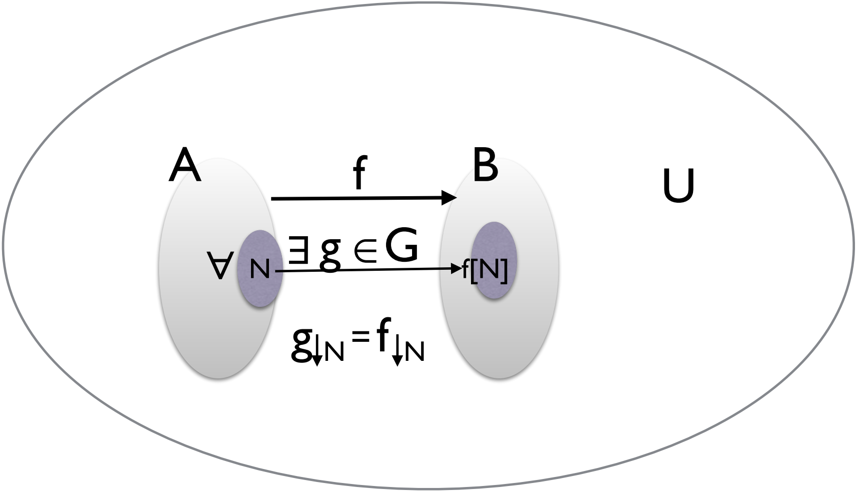
The group G has *arity* less than or equal to $n \in \omega$ if

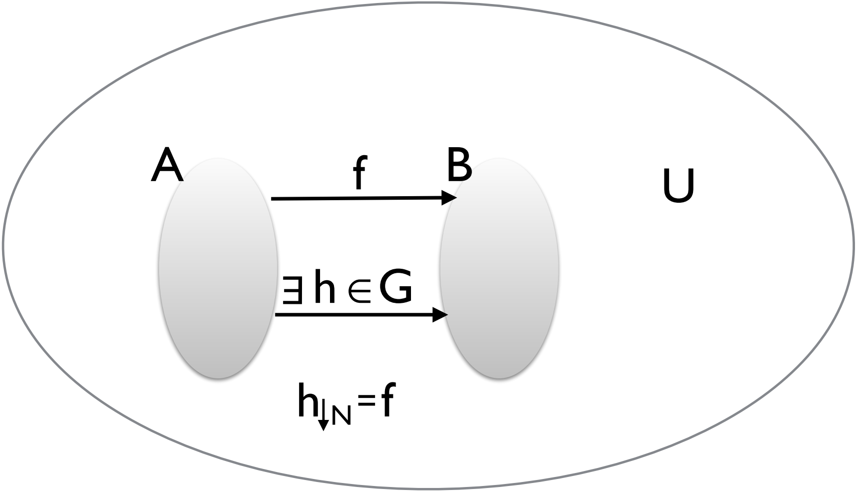
for all finite subsets A and B of U and bijections $f : A \to B$:

If there exists for every $N \in A^n$ a $g \in \mathrm{G}$ for which

f restricted to N is equal to g restricted to N

Then there is a function $h \in \mathrm{G}$ whose restriction to A equals f.


A
f
B
U
∀ N
∃ g ∈ G
f[N]
g↓N = f↓N


A
f
B
U
∃ h ∈ G
h↾N = f

The group G is *oligomorphic* if for every $n \in \omega$

the action of G on n-tuples has only finitely many

transitivity classes.

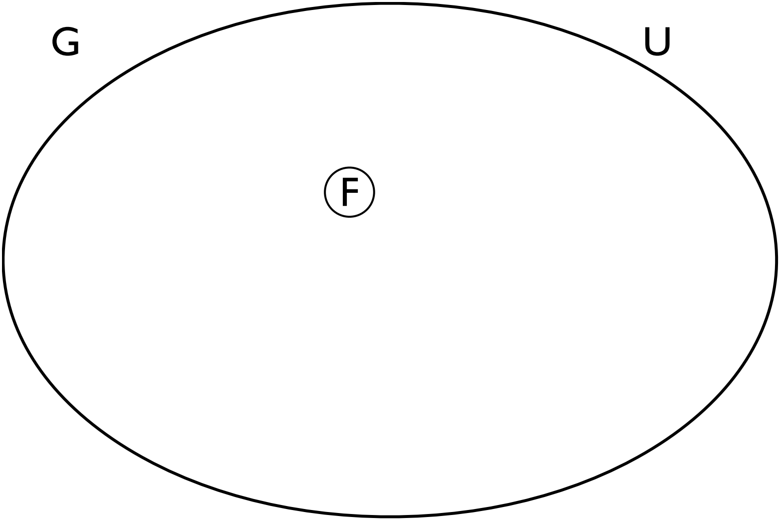
Let $F \subseteq U$ be finite. Then:

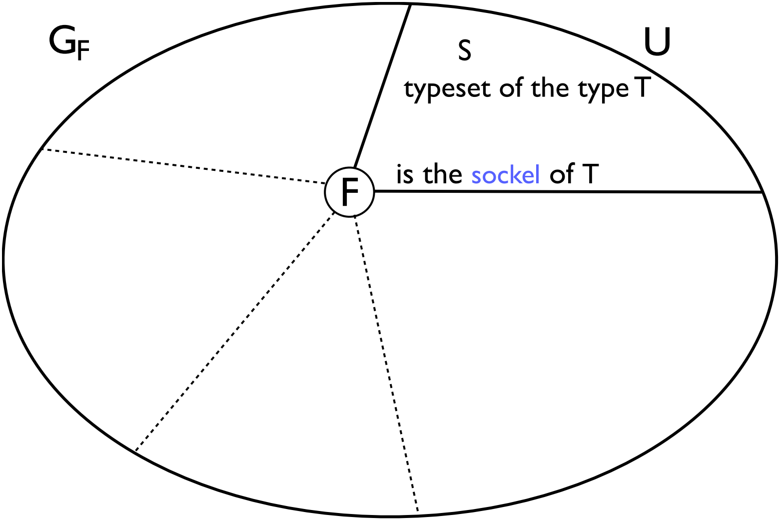
$$\mathrm{G}_F := \{f \in \mathrm{G} : \text{for all } x \in F \left(f(x) = x\right)\}.$$

A *type* is a pair consisting of a finite $F \subseteq U$

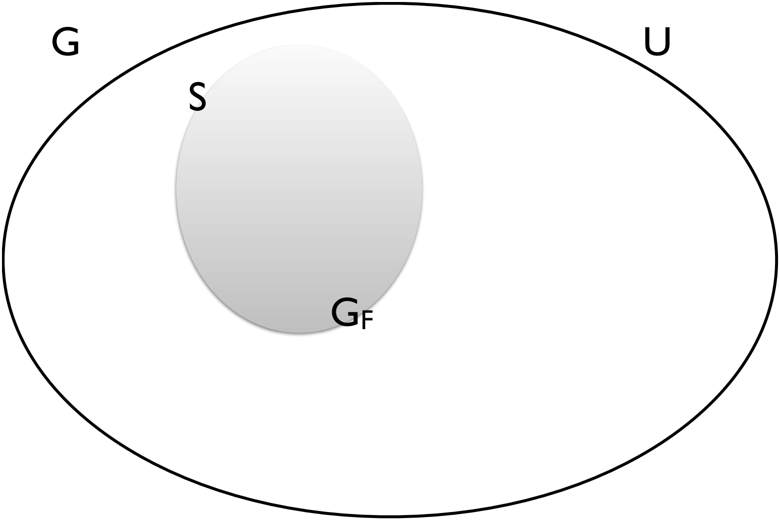
and a transitivity class of G_F

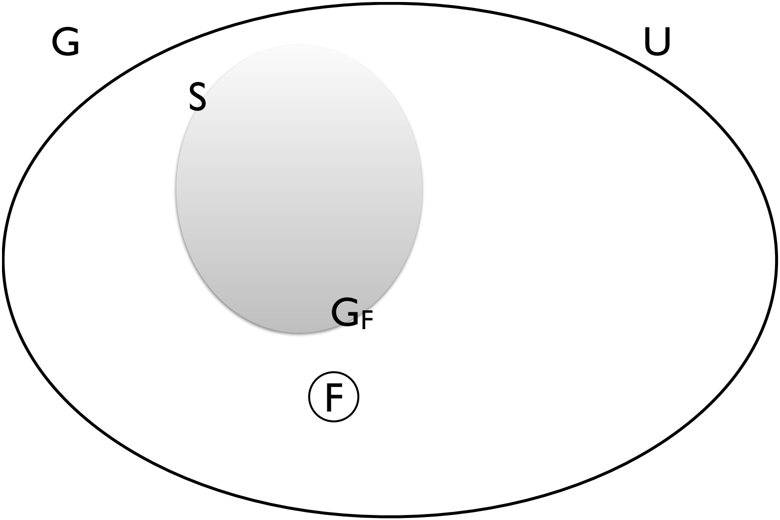
This transitivity class is the the *typeset* of the type.

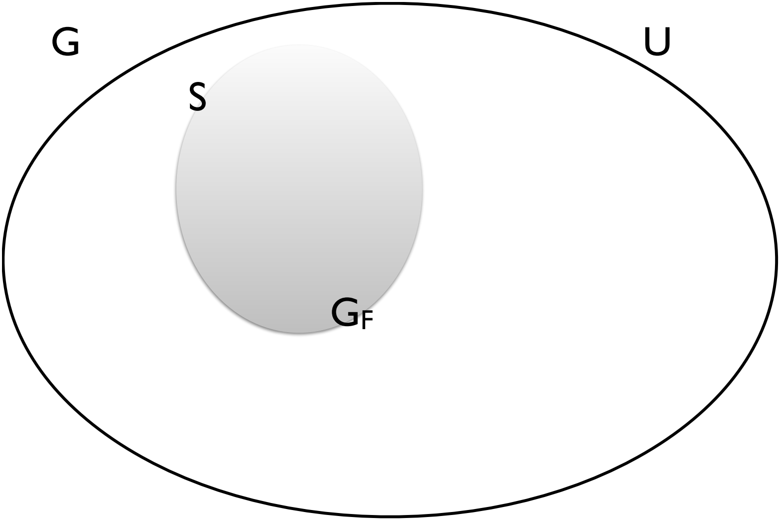

G
U
F

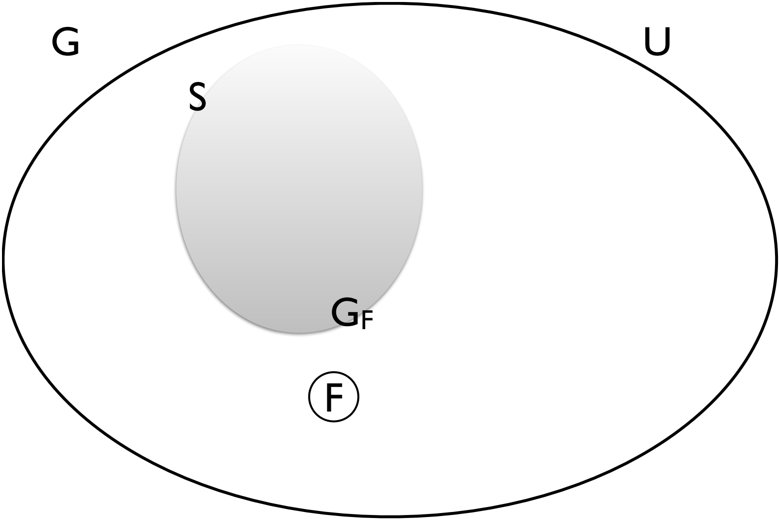


G_F
S
U
typeset of the type T
is the sockel of T
F


G
U
S
G_F


G
U
S
G_F
F


G
U
S
G_F


G
U
S
G_F
F

The group G is oligomorphic if and only if

there are only finitely many typesets for every sockel F.

The group G has *algebraic closure* if it has some finite typesets.

Let S be a subset of U.

The *age* of S is the set

$$\rho(S) := \{A \subseteq U : A \text{ is finite and there exists } f \in \mathrm{G}\,(f[A] \subseteq S)\}$$

A subset S of U is *age indivisible* if for

every partition $(P_0, P_1, \dots P_{n-1})$ of S into finitely many parts

there exists an $i \in n$ so that $\rho(P_i) = \rho(S)$.

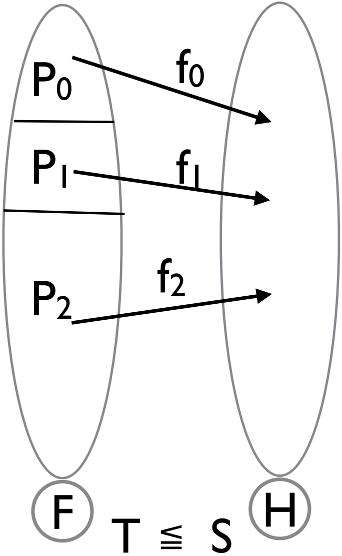
Let T and S be typesets then

$T \preceq S$ if there exists a finite partition $(P_0, P_1, \ldots, P_{n-1})$ of T

and a tuple $(f_0, f_1, \ldots, f_{n-1})$ of functions in G

so that $f_i[P_i] \subseteq S$ for all $i \in n$.

Two types are ordered under $\preceq$ if their typesets are so ordered.


P0
P1
P2
f0
f1
f2
F
H
T ≦ S

Theorem

.

If the group G *is indivisible*

then the set of infinite typesets

forms a linear quasi order ordered under $\preceq$.

Theorem

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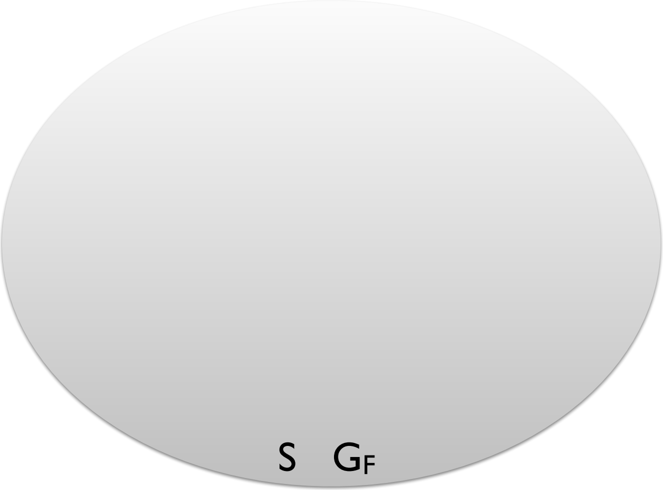
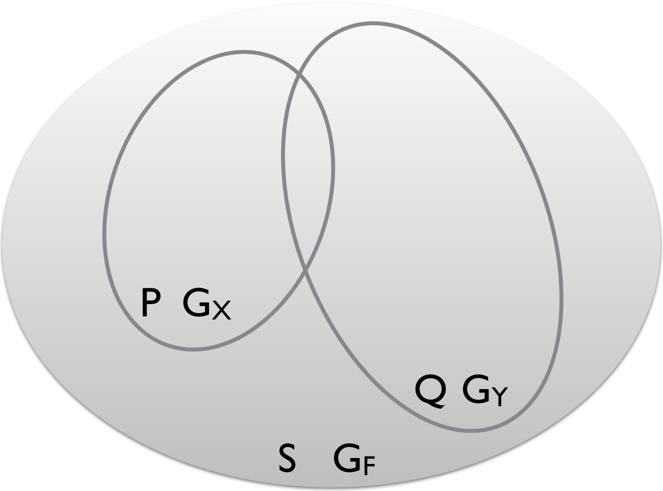
If the group G *and its typesets are indivisible*

then the set of infinite typesets

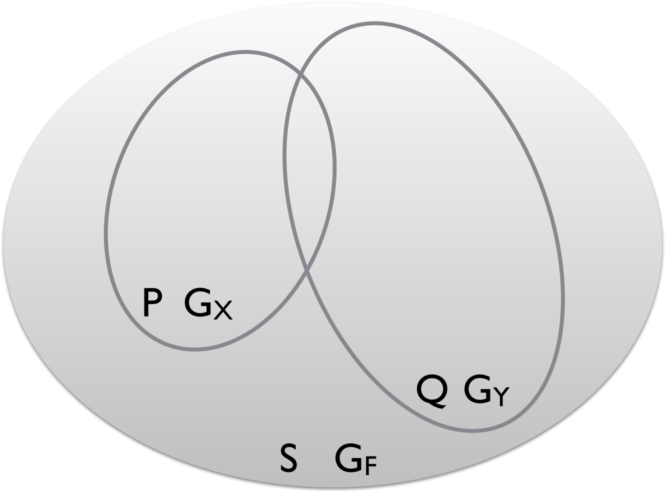
forms a linear quasi order ordered under $\subseteq$*.*

The blending property

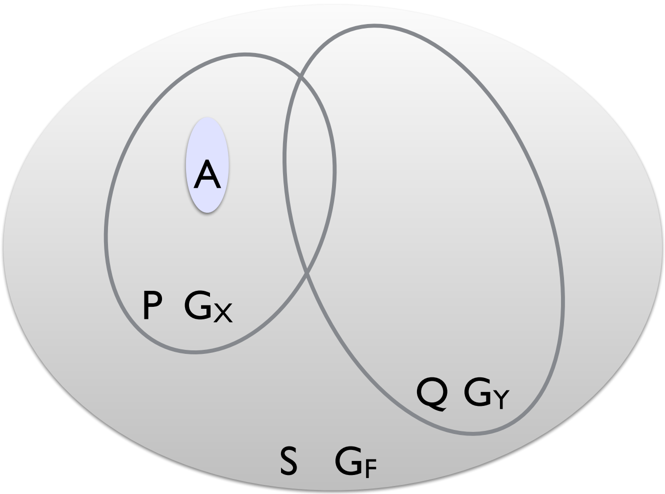

S G_F

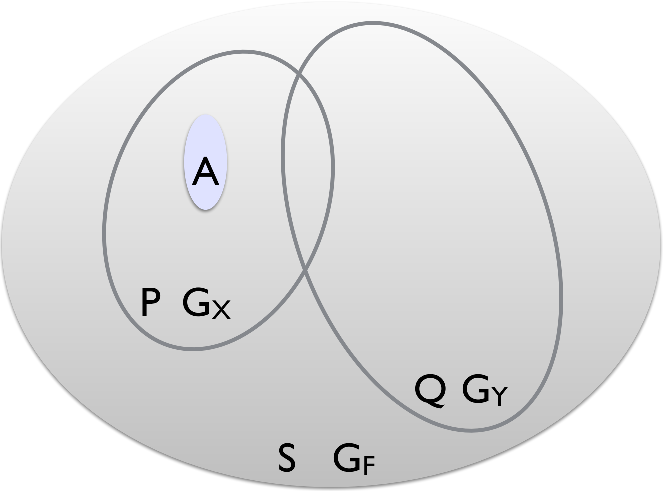


$X = F \cup \{x\}$



$\rho(P) = \rho(Q) \subsetneq \rho(S)$


A
P G_X
Q G_Y
S G_F



Then there exist functions $g \in \mathrm{G}_Y$ and $f \in \mathrm{G}_{g[X]}$ with

$(f \circ g)[A] \subseteq Q$.

Groups in which this conclusion always holds,

are the *blendable groups*.

All groups of arity two are blendable. But:

Not all groups of larger arity than two are blendable.

Theorem
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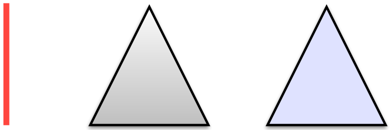
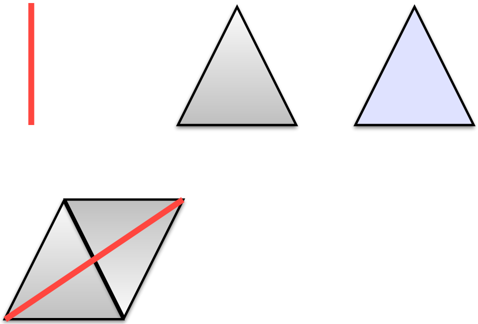
Let G be blendable and the automorphism group of a

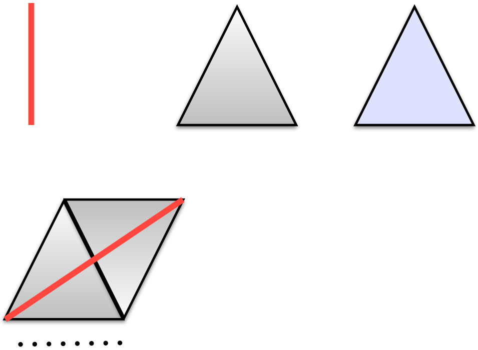
Henson type homogeneous structure.

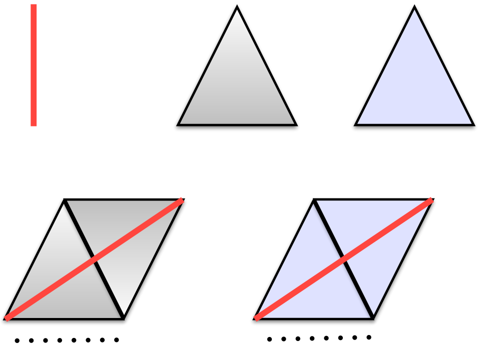
Then G is indivisible if and only if:

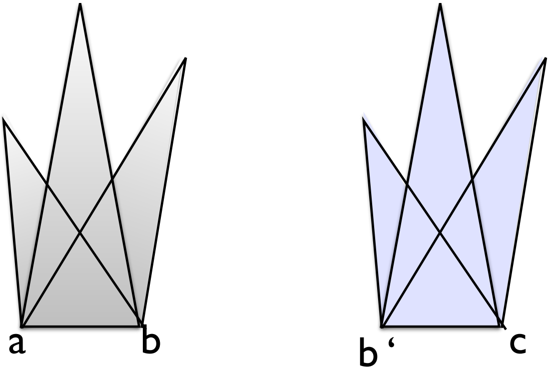
The set of ages of its typesets form a linear quasi order under $\subseteq$.

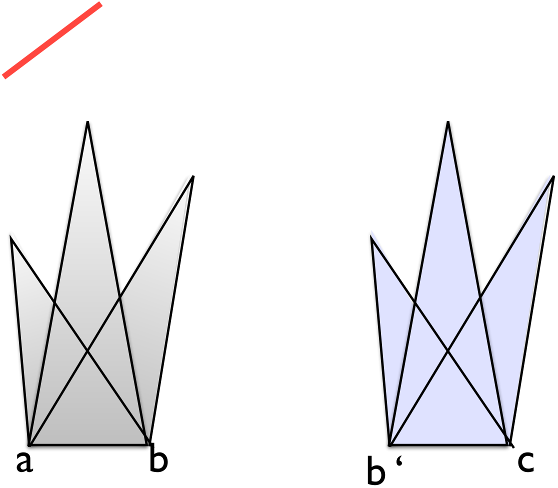


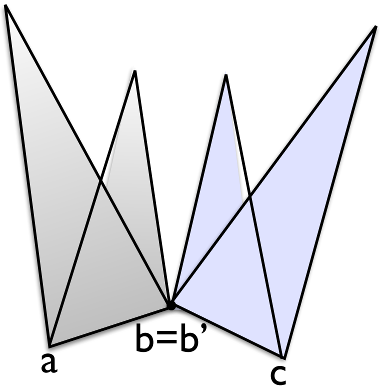


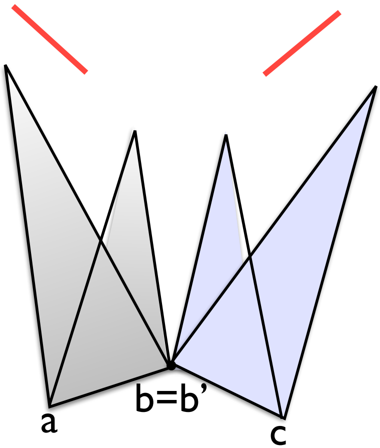


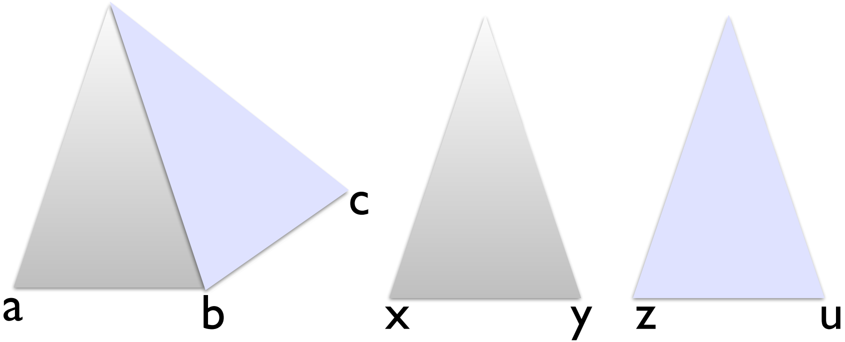


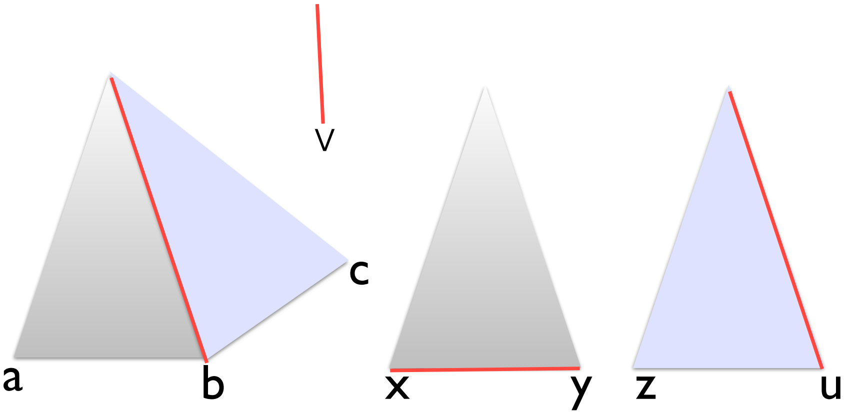

a
b
b ‘
c

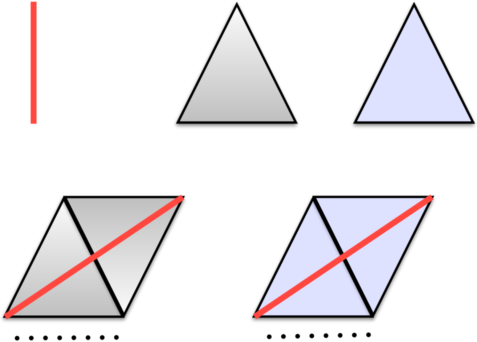

a
b
b ‘
c

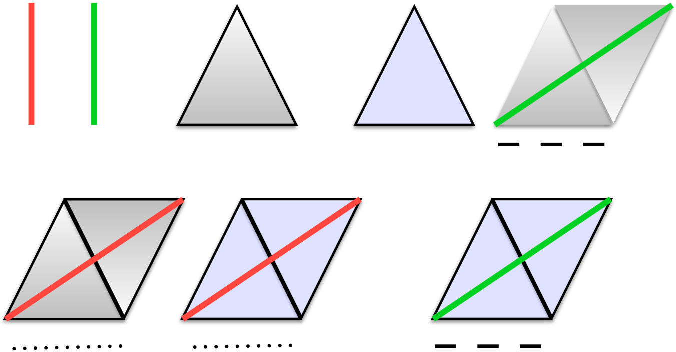

a
b=b'
c

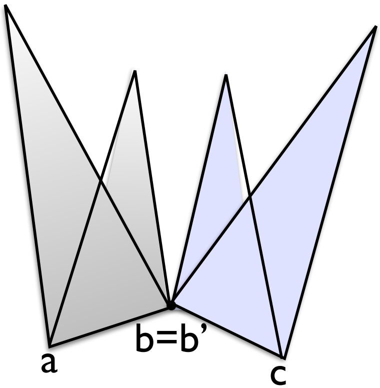

b=b'
a
c

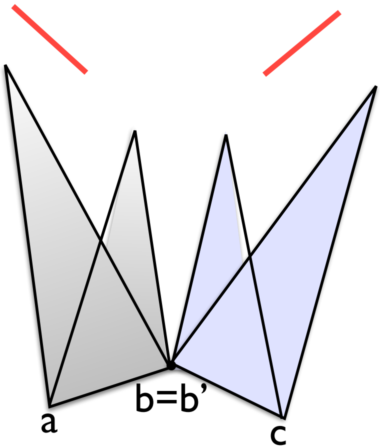

a
b
c
x
y
z
u

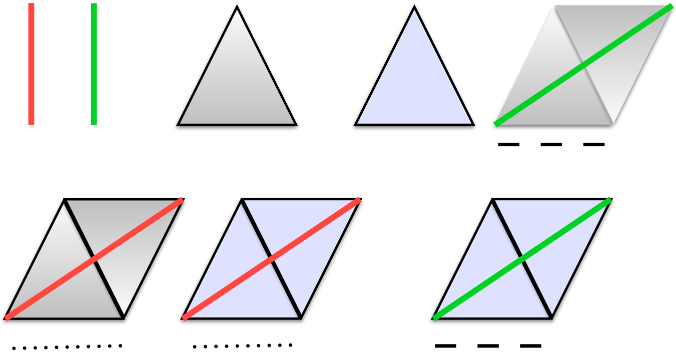

v
c
a
b
x
y
z
u

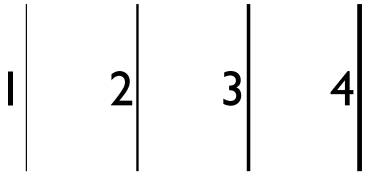


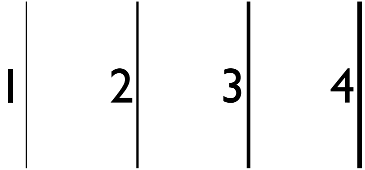


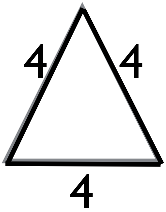

a
b=b'
c

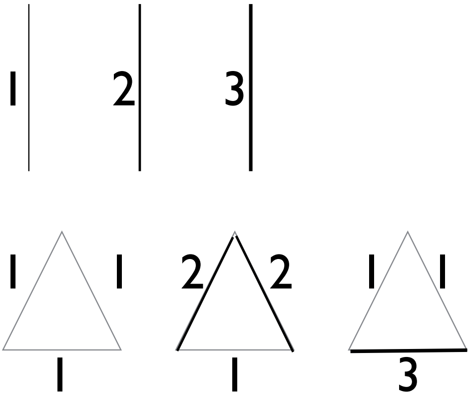

a
b=b’
c

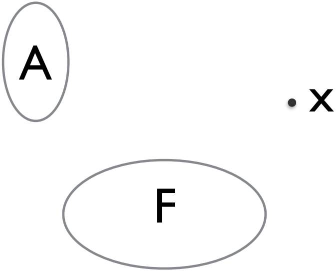



1
2
3
4


1
2
3
4

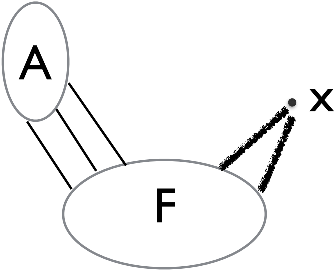

4
4
4

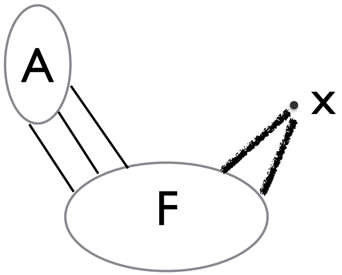

1
2
3
1 1 1
2 2 1
1 1 3

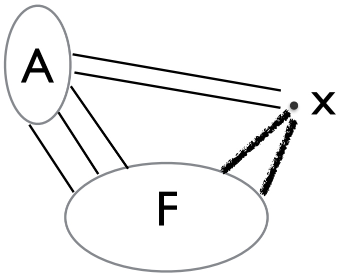


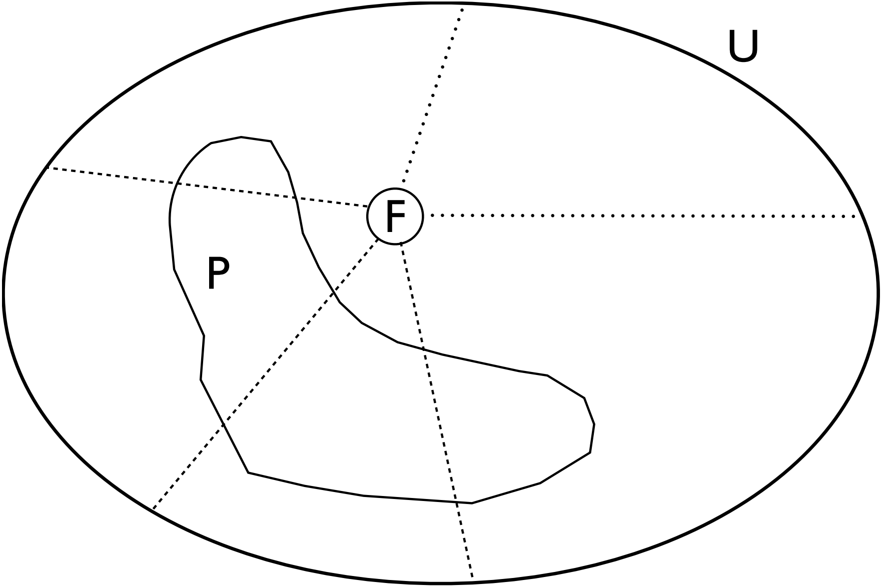
The finite sets $A \subseteq U$ and $F \subseteq U$ are sucht that A is a subset of a typeset with sockel F. Also $x \in U$.

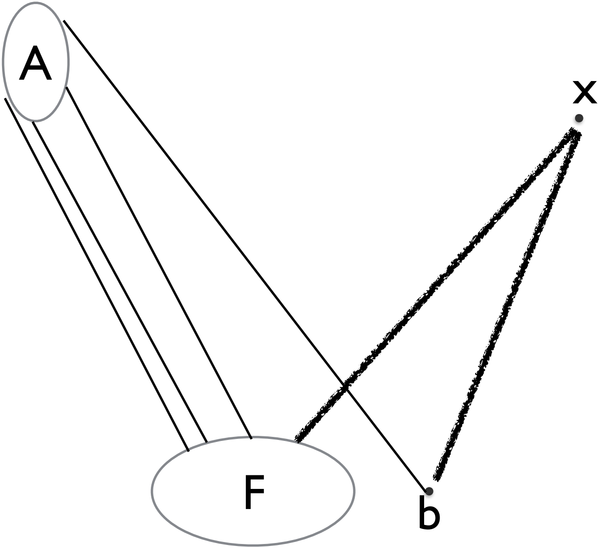
A is *consistently placed with respect to F*.


A
F
x


A
F
x


A
F
x


U
F
P


A
x
F
b