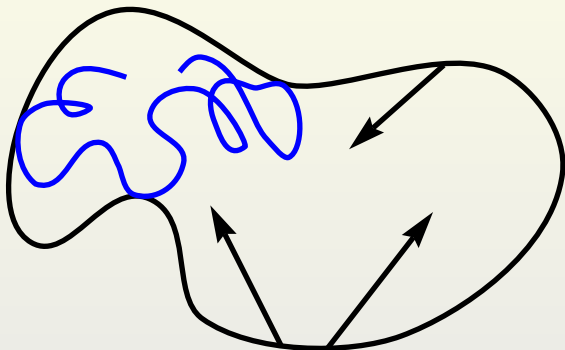


REFLECTED BROWNIAN MOTION — DEFINITION AND APPROXIMATION

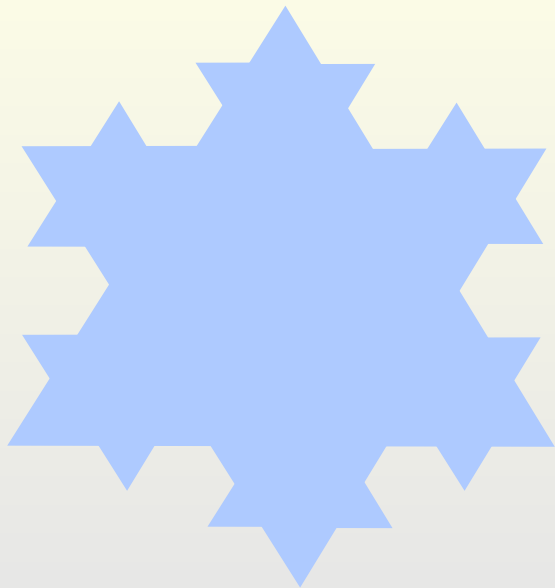
Krzysztof Burdzy
University of Washington

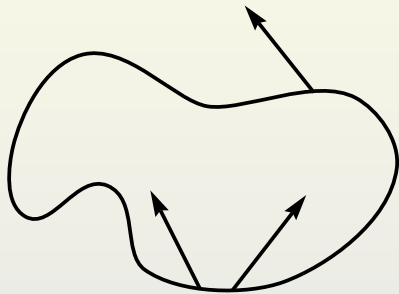
Zhenqing Chen, Donald Marshall, Kavita Ramanan

Obliquely reflected Brownian motion

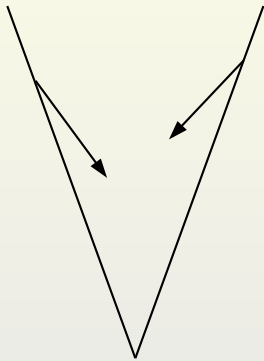
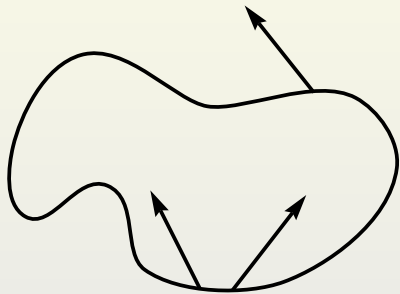


Obliquely reflected Brownian motion in fractal domains





Technical challenges



Classical Dirichlet form approach to Markov processes is limited to symmetric processes. Obliquely reflected Brownian motion is not symmetric.

Classical Dirichlet form approach to Markov processes is limited to symmetric processes. Obliquely reflected Brownian motion is not symmetric.

Non-symmetric Dirichlet form approach to obliquely reflected Brownian motion had limited success (Kim, Kim and Yun (1998) and Duarte (2012)).

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

THEOREM (B and Marshall; 1993)

For an arbitrary measurable θ , obliquely reflected Brownian motion X in D with the oblique angle of reflection θ exists.

D – unit disc in \mathbb{R}^2

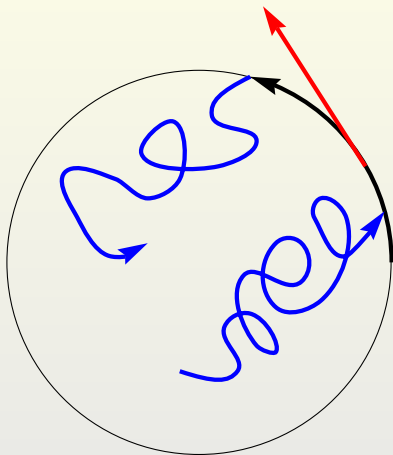
$\theta(x)$ – angle of reflection at $x \in \partial D$

THEOREM (B and Marshall; 1993)

For an arbitrary measurable θ , obliquely reflected Brownian motion X in D with the oblique angle of reflection θ exists.

Lions and Sznitman (1984), Harrison, Landau and Shepp (1985),
Varadhan and Williams (1985)

Jumps on the boundary



Parametrization of obliquely reflected Brownian motions

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

Parametrization of obliquely reflected Brownian motions

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

$$\theta \leftrightarrow (h, \mu)$$

Parametrization of obliquely reflected Brownian motions

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

$$\theta \leftrightarrow (h, \mu)$$

$h(x)dx$ – stationary distribution

Parametrization of obliquely reflected Brownian motions

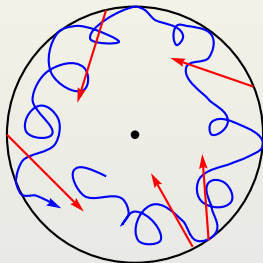
D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

$$\theta \leftrightarrow (h, \mu)$$

$h(x)dx$ – stationary distribution

μ – rate of rotation



Parametrization of obliquely reflected Brownian motions

D – unit disc in \mathbb{R}^2

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

$$h(z) = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \Re(e^{-i\theta(0)})} = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \cos \theta(0)}, \quad z \in D,$$

$$\mu = \tan \theta(0) = \int_D \tan \theta(z) h(z) dz,$$

$$\theta(z) = -\arg \left(h(z) + i\tilde{h}(z) - i\mu/\pi \right), \quad z \in D.$$

Parametrization of obliquely reflected Brownian motions

D – unit disc in \mathbb{R}^2

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

$$h(z) = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \Re(e^{-i\theta(0)})} = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \cos \theta(0)}, \quad z \in D,$$

$$\mu = \tan \theta(0) = \int_D \tan \theta(z) h(z) dz,$$

$$\theta(z) = -\arg \left(h(z) + i\tilde{h}(z) - i\mu/\pi \right), \quad z \in D.$$

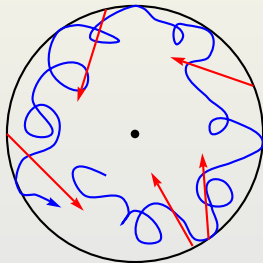
Harrison, Landau and Shepp (1985): smooth θ

Rate of rotation of obliquely reflected Brownian motion

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

- ① Assume that θ is C^2 . Then, with probability 1, X is continuous and, therefore, $\arg X_t$ is well defined for $t > 0$. The distributions of $\frac{1}{t} \arg X_t - \mu$ converge to the Cauchy distribution when $t \rightarrow \infty$.
- ② Let $\arg^* X_t$ be $\arg X_t$ “without large excursions from ∂D .” Then, a.s.,

$$\lim_{t \rightarrow \infty} \arg^* X_t / t = \mu.$$



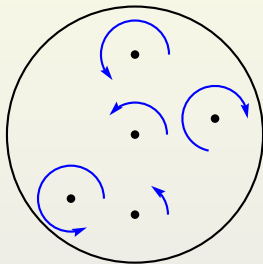
D – simply connected bounded open set in \mathbb{R}^2

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

For every positive harmonic function h in D with L^1 norm equal to 1 and every real number μ , there exists a (unique in distribution) obliquely reflected Brownian motion in D with the stationary distribution $h(x)dx$ and rate of rotation μ .

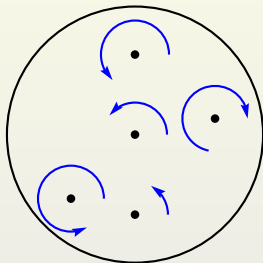
Rotation rate field

Let D be the unit disc and $\mu(z)$ be the rate of rotation around $z \in D$. In other words, the distributions of $\frac{1}{t} \arg(X_t - z) - \mu(z)$ converge to the Cauchy distribution when $t \rightarrow \infty$.



Rotation rate field

Let D be the unit disc and $\mu(z)$ be the rate of rotation around $z \in D$. In other words, the distributions of $\frac{1}{t} \arg(X_t - z) - \mu(z)$ converge to the Cauchy distribution when $t \rightarrow \infty$.



THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

The function $\mu(z)$ is harmonic in D .

Rotation rate field

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

$h(x)dx$ – stationary distribution

$\mu(z)$ – rate of rotation around z

$$\theta \leftrightarrow (h, \mu(0)) \leftrightarrow \{\mu(z)\}_{z \in D}$$

Rotation rate field

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

$h(x)dx$ – stationary distribution

$\mu(z)$ – rate of rotation around z

$$\theta \leftrightarrow (h, \mu(0)) \leftrightarrow \{\mu(z)\}_{z \in D}$$

Arrows indicate one to one mappings. Are the mappings surjective?

Rotation rate field

D – unit disc in \mathbb{R}^2

$\theta(x)$ – angle of reflection at $x \in \partial D$

$h(x)dx$ – stationary distribution

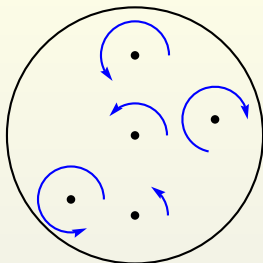
$\mu(z)$ – rate of rotation around z

$$\theta \leftrightarrow (h, \mu(0)) \leftrightarrow \{\mu(z)\}_{z \in D}$$

Arrows indicate one to one mappings. Are the mappings surjective?

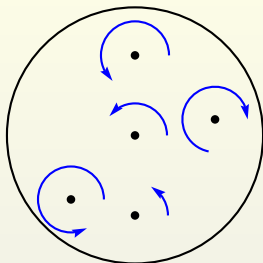
In the first case, yes.

Rotation rate field – limitations



Suppose that $\phi(z)$ is harmonic in the unit disc D . $\phi(z)$ does not have to be positive.

Rotation rate field – limitations

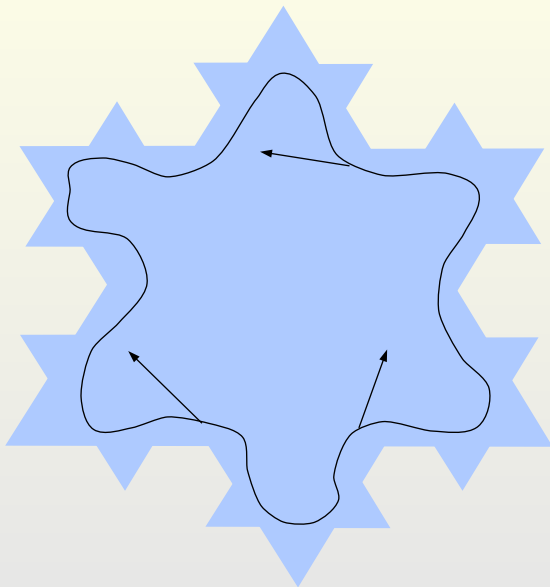


Suppose that $\phi(z)$ is harmonic in the unit disc D . $\phi(z)$ does not have to be positive.

For each ϕ there exists $b_0 \geq 0$ such that for all $a \in \mathbb{R}$ and $b \in [0, b_0]$, the function $\mu(z) = a + b\phi(z)$ is the rotation field of an obliquely reflected Brownian motion in D .

For $b > b_0$, the function $\mu(z) = a + b\phi(z)$ does not represent the rotation field of an obliquely reflected Brownian motion in D .

Obliquely reflected Brownian motion in fractal domains



Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

$\theta_k(x)$ – reflection angle; $x \in \partial D_k$

X^k – obliquely reflected Brownian motion in D_k

Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

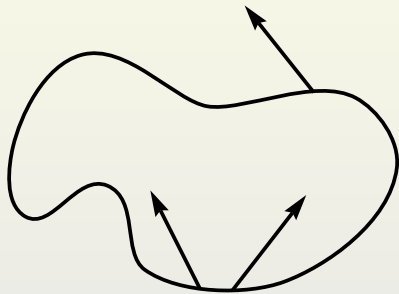
$\theta_k(x)$ – reflection angle; $x \in \partial D_k$

X^k – obliquely reflected Brownian motion in D_k

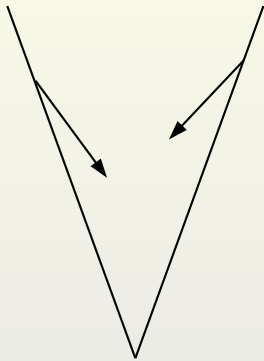
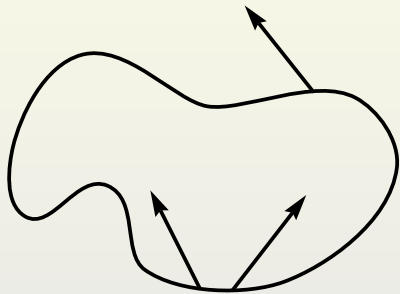
THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

Suppose that θ_k converge as $k \rightarrow \infty$. Then obliquely reflected Brownian motions X^k converge, as $k \rightarrow \infty$, to a process in D .

We apply conformal invariance of Brownian motion.



Technical challenges



Integrability of harmonic functions

Let $\delta_D(x) = \text{dist}(x, \partial D)$ and $x_0 \in D$. We say that D is a John domain with John constant $c_J > 0$ if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that $\delta_D(y) \geq c_J \ell(\gamma(x, y))$ for all $y \in \gamma$, where $\gamma(x, y)$ is the subarc of γ from x to y and $\ell(\gamma(x, y))$ is the length of $\gamma(x, y)$.

THEOREM (Aikawa, 2000)

- (i) If $D \subset \mathbb{R}^2$ is a bounded John domain with John constant $c_J \geq 7/8$ then all positive harmonic functions in D are in $L^1(D)$.
- (ii) If $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain with constant $\lambda < 1$ then all positive harmonic functions in D are in $L^1(D)$.
- (iii) There exists a bounded Lipschitz domain D with constant $\lambda = 1$ and a positive harmonic function h in D which is not in $L^1(D)$.

The modulus of continuity of f is $\omega_f(a) = \sup_{|x-y|<a} |f(x) - f(y)|$ and f is Dini continuous if $\int_0^b (\omega_f(a)/a) da < \infty$ for some $b > 0$.

THEOREM

- (i) (Garnett, 2007) If θ is Dini continuous then h is bounded.
- (ii) Suppose that ω is an increasing continuous concave function on $[0, \pi/2]$ such that $\omega(0) = 0$, $\omega(\pi/2) = \pi/4$, and $\int_0^{\pi/2} (\omega(a)/a) da = \infty$. Then there exists θ such that $\omega_\theta(a) = \omega(a)$ for $a \leq \pi/2$ and h is unbounded.

Myopic conditioning

$D \subset \mathbb{R}^d$ – open bounded connected set

$\varepsilon > 0$

X_t^ε – a continuous process in D

DEFINITION (Myopic Brownian motion)

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c (during the time interval $[k\varepsilon, (k+1)\varepsilon]$).

Myopic conditioning

$D \subset \mathbb{R}^d$ – open bounded connected set

$\varepsilon > 0$

X_t^ε – a continuous process in D

DEFINITION (Myopic Brownian motion)

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c (during the time interval $[k\varepsilon, (k+1)\varepsilon]$).

THEOREM (B, Chen)

Processes X^ε converge weakly, as $\varepsilon \rightarrow 0$, to reflected Brownian motion in D .

Myopic conditioning (2)

$D \subset \mathbb{R}^d$ – open bounded connected set, $\varepsilon > 0$

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c during the time interval $[k\varepsilon, (k+1)\varepsilon]$.

Myopic conditioning (2)

$D \subset \mathbb{R}^d$ – open bounded connected set, $\varepsilon > 0$

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c during the time interval $[k\varepsilon, (k+1)\varepsilon]$.

B – Brownian motion in \mathbb{R}^d , $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$

$$Y_k^\varepsilon = X_{k\varepsilon}^\varepsilon, \quad k \geq 1$$

$$m_\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$$

Myopic conditioning (2)

$D \subset \mathbb{R}^d$ – open bounded connected set, $\varepsilon > 0$

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c during the time interval $[k\varepsilon, (k+1)\varepsilon]$.

B – Brownian motion in \mathbb{R}^d , $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$

$$Y_k^\varepsilon = X_{k\varepsilon}^\varepsilon, \quad k \geq 1$$

$$m_\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$$

LEMMA (B, Chen)

(i) $m_\varepsilon \rightarrow$ Lebesgue measure on D as $\varepsilon \rightarrow 0$.

(ii) $m_\varepsilon(dx)$ is a reversible (stationary) measure for Y_k^ε .

Increasing families of domains

$D \subset \mathbb{R}^d$ – open bounded connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

Increasing families of domains

$D \subset \mathbb{R}^d$ – open bounded connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

X^k – reflected Brownian motion in D_k

Increasing families of domains

$D \subset \mathbb{R}^d$ – open bounded connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

X^k – reflected Brownian motion in D_k

THEOREM (B, Chen; 1998)

Reflected Brownian motions X^k converge, as $k \rightarrow \infty$, to reflected Brownian motion in D .

Invariance principle for reflected random walks

D – open connected bounded set

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

Invariance principle for reflected random walks

D – open connected bounded set

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

Assume that D is an extension domain. Then reflected random walks X^k , with sped-up clocks, converge weakly to reflected Brownian motion in D , as $k \rightarrow \infty$.

Invariance principle for reflected random walks

D – open connected bounded set

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

Assume that D is an extension domain. Then reflected random walks X^k , with sped-up clocks, converge weakly to reflected Brownian motion in D , as $k \rightarrow \infty$.

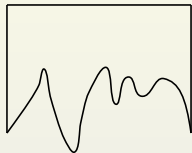
Examples of extension domains.

- ① Smooth domains
- ② Lipschitz domains
- ③ Uniform domains
- ④ NTA domains
- ⑤ Von Koch snowflake

Invariance principle in domains above graphs of continuous functions

D – bounded domain

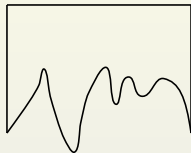
∂D is locally the graph of a continuous function



Invariance principle in domains above graphs of continuous functions

D – bounded domain

∂D is locally the graph of a continuous function



Fact: D is an extension domain.

COROLLARY (B, Chen; 2008)

Assume that D lies locally above the graph of a continuous function. Then reflected random walks X^k , with sped-up clocks, converge weakly to reflected Brownian motion in D , as $k \rightarrow \infty$.

Invariance principle – a counterexample

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

Invariance principle – a counterexample

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks X^k , with sped-up clocks, do not converge weakly to reflected Brownian motion in D , when $k \rightarrow \infty$.

Invariance principle – a counterexample

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks X^k , with sped-up clocks, do not converge weakly to reflected Brownian motion in D , when $k \rightarrow \infty$.

Example: Remove suitable dust from a square.

Invariance principle (improved)

D – open connected bounded set

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

Assume that D is an extension domain. Then reflected random walks X^k , with sped-up clocks, converge weakly to reflected Brownian motion in D .

Invariance principle (improved)

D – open connected bounded set

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

Assume that D is an extension domain. Then reflected random walks X^k , with sped-up clocks, converge weakly to reflected Brownian motion in D .

D – open connected bounded set

D_k – subset of $D \cap (2^{-k}\mathbb{Z}^2)$; contains all vertices of the union of adjacent cubes in D

X^k – reflected random walk on D_k

X^k can jump along an edge if the edge is in D_k

Invariance principle (improved)

D – open connected bounded set

X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

X^k can jump along an edge if the edge is in D

THEOREM (B, Chen; 2008)

Assume that D is an extension domain. Then reflected random walks X^k , with sped-up clocks, converge weakly to reflected Brownian motion in D .

D – open connected bounded set

D_k – subset of $D \cap (2^{-k}\mathbb{Z}^2)$; contains all vertices of the union of adjacent cubes in D

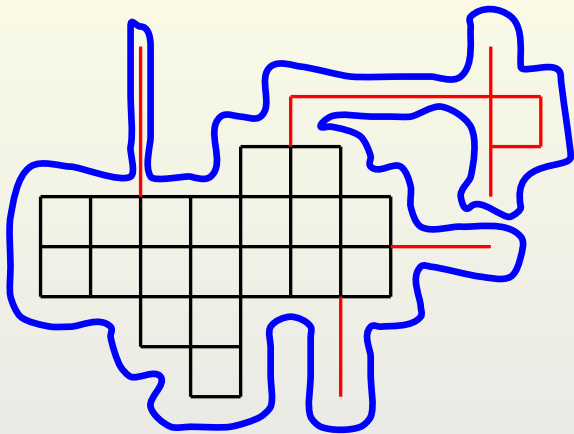
X^k – reflected random walk on D_k

X^k can jump along an edge if the edge is in D_k

THEOREM (B, Chen; 2012)

Reflected random walks X^k on D_k , with sped-up clocks, converge weakly to reflected Brownian motion in D , as $k \rightarrow \infty$.

Two approximations



THEOREM (B, Chen; 2012)

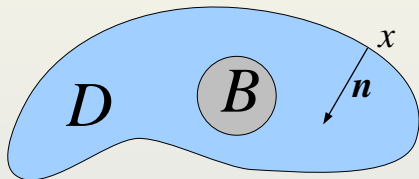
Suppose that $D \subset \mathbb{R}^d$ is a domain with finite volume. There exists a countable sequence of bounded functions $\{\varphi_j\}_{j \geq 1} \subset W^{1,2}(D) \cap C^\infty(D)$ such that

- ① $\{\varphi_j\}_{j \geq 1}$ is dense in $W^{1,2}(D)$,
- ② $\{\varphi_j\}_{j \geq 1}$ separates points in D ,
- ③ for each $j \geq 1$,

$$\limsup_{k \rightarrow \infty} 2^{k(2-d)} \sum_{\overline{xy} \in D_k} (\varphi_j(x) - \varphi_j(y))^2 \leq 2 \int_D |\nabla \varphi_j(x)|^2 dx.$$

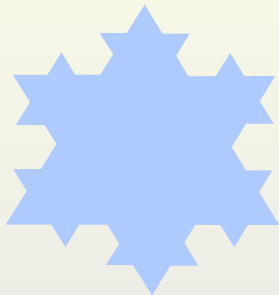
Robin problem

$$\begin{aligned}\Delta u(x) &= 0, & x \in D \setminus B, \\ \frac{\partial u}{\partial \mathbf{n}}(x) &= cu(x), & x \in \partial D, \\ u(x) &= 1, & x \in \partial B.\end{aligned}$$



Robin problem in fractal domains

Example: von Koch snowflake.



The normal vector does not exist at almost all boundary points.

Naive approach

Approximate the snowflake domain D with an increasing sequence of smooth domains D_k , such that $\bigcup_k D_k = D$.

Approximate the snowflake domain D with an increasing sequence of smooth domains D_k , such that $\bigcup_k D_k = D$.

Let u_k be the solution to the Robin boundary problem in D_k , with the same c (adsorption rate) for all k , and let

$$u(x) = \lim_{k \rightarrow \infty} u_k(x).$$

Approximate the snowflake domain D with an increasing sequence of smooth domains D_k , such that $\bigcup_k D_k = D$.

Let u_k be the solution to the Robin boundary problem in D_k , with the same c (adsorption rate) for all k , and let

$$u(x) = \lim_{k \rightarrow \infty} u_k(x).$$

Then u satisfies the Dirichlet boundary conditions $u(x) = 0$ on ∂D .

Reformulation of Robin problem

Assuming that D is smooth, the Green-Gauss formula implies that for $u, v \in C^2(\overline{D})$,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_D v(x) \Delta u(x) dx - \int_{\partial D} v(x) \frac{\partial u}{\partial \mathbf{n}}(x) \sigma(dx),$$

where σ is the surface measure on ∂D .

Reformulation of Robin problem

Assuming that D is smooth, the Green-Gauss formula implies that for $u, v \in C^2(\overline{D})$,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_D v(x) \Delta u(x) dx - \int_{\partial D} v(x) \frac{\partial u}{\partial \mathbf{n}}(x) \sigma(dx),$$

where σ is the surface measure on ∂D .

A weak solution u to the Robin problem is characterized by

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_{\partial D} cu(x)v(x)\sigma(dx),$$

for every $v \in C^2(\overline{D})$ that vanishes on B .

Solution to Robin problem in von Koch snowflake

$$d = \log 4 / \log 3$$

Let μ be d -dimensional Hausdorff measure.

Solution to Robin problem in von Koch snowflake

$$d = \log 4 / \log 3$$

Let μ be d -dimensional Hausdorff measure.

DEFINITION

We will say that a function u is a weak solution to the Robin problem in the snowflake domain if for all smooth v ,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_{\partial D} cu(x)v(x)\mu(dx).$$

Alternative representation

D – von Koch snowflake domain

X – reflected Brownian motion in D

σ_B – hitting time of B

Alternative representation

D – von Koch snowflake domain

X – reflected Brownian motion in D

σ_B – hitting time of B

L – “local time” on ∂D , i.e., a continuous additive functional of X with Revuz measure μ

Alternative representation

D – von Koch snowflake domain

X – reflected Brownian motion in D

σ_B – hitting time of B

L – “local time” on ∂D , i.e., a continuous additive functional of X with Revuz measure μ

THEOREM (forthcoming; B, Chen)

- The continuous additive functional L with Revuz measure μ exists.

Alternative representation

D – von Koch snowflake domain

X – reflected Brownian motion in D

σ_B – hitting time of B

L – “local time” on ∂D , i.e., a continuous additive functional of X with Revuz measure μ

THEOREM (forthcoming; B, Chen)

- The continuous additive functional L with Revuz measure μ exists.
- The function

$$u(x) = \mathbb{E}_x \left[\exp \left(-\frac{c}{2} \int_0^{\sigma_B} dL_s \right) \right], \quad x \in \overline{D} \setminus B,$$

is the unique weak solution to the Robin problem.