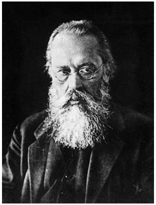

Steklov spectral inequalities through quasiconformal mapping

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I. Survey of the Steklov eigenvalue problem

[Girouard & Polterovich *Spectral geometry of the Steklov problem*, 2014]

Vibrating simply conn. free membrane $\Omega \subset \mathbb{R}^2$, mass concentrated on boundary. Transverse displacement $u(x, y)$ has Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^2 \, dA}{\int_{\partial\Omega} u^2 \, ds}$$

where ds =arclength element.

Euler-Lagrange gives **Steklov eigenvalue problem**

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega \end{cases}$$

Eigenfunction u is harmonic, and Steklov eigenvalues are

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty.$$

Usual variational characterization of σ_j via Rayleigh quotient. Note $u_0 \equiv$ const.

Example — unit disk

$u = r^k \cos k\theta$ and $u = r^k \sin k\theta$ e.g. $k = 2$

$$\frac{\partial u}{\partial n} = ku \quad \Longrightarrow \quad \text{spectrum } \{0, 1, 1, 2, 2, 3, 3, \ldots\}$$

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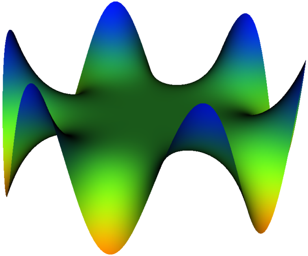
e.g. $k = 6$, whispering gallery:

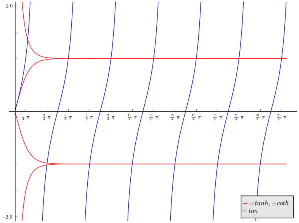


Example — square $(-1,1) \times (-1,1)$

Separation of variables: find $u = xy$ with $\sigma = 1$. Also
$u = \sin(\alpha x)\sinh(\alpha y)$, where α is root of $\boxed{\tan\alpha = \tanh\alpha}$

$$\implies \quad \text{eigenvalue} \quad \sigma = \alpha\coth\alpha.$$

Find cos, cosh solutions too. In plot below (from G&P 2014), each intersection gives a double eigenvalue:



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$\implies$ eigenvalue $\sigma = \alpha \coth \alpha$.

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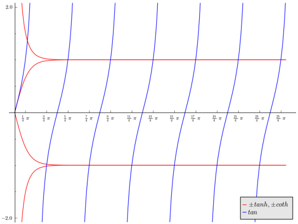


How prove *completeness* of eigenfunction collection?!
By reducing to 1-d beam equation on boundary edge.

Rectangle — *completeness* of the separated eigenfunctions is a open problem; work in progress by Girouard and Polterovich.

Weyl asymptotics

For smooth boundary,

$$\boxed{\sigma_j L = j\pi + O(1)} \qquad \text{as } j \to \infty$$

where L is length of $\partial\Omega$.

Proof: pseudo differential operator techniques for Dirichlet-to-Neumann operator.

Improved error estimate by Rozenbljum says eigenvalues come in pairs, and converge onto disk eigenvalues faster than any power of j:

$$\sigma_{2j} L = 2j\pi + O(j^{-\infty})$$
$$\sigma_{2j-1} L = 2j\pi + O(j^{-\infty})$$

Open problem: prove Weyl asymptotic $\sigma_j L = j\pi + O(1)$ assuming only a Lipschitz boundary.

Inverse spectral problem

Mark Kac

Can you hear the shape of a Steklov membrane?

No known counterexamples.
Cut-and-paste constructions (used for Laplace eigenvalues) fail since the eigenvalue appears in the boundary condition!

What *can* be heard?

Spectrum determines boundary length (by Weyl),
number of boundary components (by Girouard *et al.* 2014).

II. Spectral inequalities — lower bounds

(a) Length normalization is not enough:

$\sigma_j L$ can be arbitrarily close to 0 for thin rectangle $(0, \pi) \times (0, \epsilon)$.

Intuition: area vanishes as $\epsilon \to 0$, but boundary length does not.

More precisely, use $u(x, y) = \sin(kx)$ for $k = 1, 2, 3, \ldots$ as boundary-orthogonal trial functions in Rayleigh quotient. Get $\sigma_{k-1} \leq k^2 \epsilon / 2$.

(b) So introduce some geometric information...

Payne (1970) proved for convex Ω that

$$\sigma_1 \geq (\text{min. curvature on } \partial\Omega).$$

III. Spectral inequalities — upper bounds by conformal mapping

First eigenvalue by Weinstock (1954):

$$\boxed{\sigma_1 L \leq 2\pi}$$

Maximizer is disk.

Higher eigenvalues by Hersch–Payne–Schiffer (1975):

$$\boxed{\sigma_j L \leq 2\pi j}$$

Maximizer is disjoint union of *j* disks (limiting case), proved Girouard–Polterovich (2010).

Goal: want better bounds for domains that are **not** close to union of disjoint disks.

Eigenvalue estimates by *quasi*conformal maps

Theorem (Girouard, Laugesen & Siudeja, 2014)

Assume $f : \mathbb{D} \to \Omega$ is quasiconformal with complex dilatation μ depending only on the angular variable θ. Then

$$\sum_{j=1}^{n} \sigma_j L \leq \sum_{j=1}^{n} \left\lceil \frac{j}{2} \right\rceil 2\pi\, g$$

for each $n \in \mathbb{N}$, with equality if Ω is a disk.
The geometric factor $g = g(f)$ is explicitly computable.

Intuition: $g \geq 1$ with equality for disk, and so

g penalizes deviation from disk

Questions... Why care about eigenvalue sums?
How is the theorem proved? What does it say for conformal maps? for starlike domains? How does it compare with Hersch–Payne–Schiffer?

The power of eigenvalue sums

Corollary

The following spectral functionals attain their maximum when Ω is a disk:

$$\sigma_1 L/g, \quad (\sigma_1^s + \cdots + \sigma_n^s)^{1/s} L/g, \quad \sqrt[n]{\sigma_1 \cdots \sigma_n}\, L/g$$

for each $0 < s \leq 1$. Further, for $s < 0 < t$ the partial sums of the spectral zeta function and heat trace, namely

$$\sum_{j=1}^{n} (\sigma_j L/g)^s \qquad \text{and} \qquad \sum_{j=1}^{n} \exp(-t\sigma_j L/g),$$

are minimal when Ω is a disk.

Proof: Theorem + Hardy–Littlewood–Pólya majorization

Proof of Theorem

• Obtain *trial functions* on Ω by transplanting trigonometric eigenfunctions from disk, by quasiconformal map f

• Impose *orthogonality* of trial functions on boundary of Ω by "angular uniformization" of the circle

• Apply *Rayleigh variational characterization for eigenvalue sum* (taking minimum over orthogonal trial functions)

• Use quasiconformal transformation formula for *Dirichlet integral*

$$\int_{\Omega} |\nabla(u \circ f^{-1})|^2 \, dA = \int_{\mathbb{D}} \left\{ a_0 u_r^2 + a_1 \frac{u_\theta^2}{r^2} + a_2 u_r \frac{u_\theta}{r} \right\} r dr d\theta$$

where coefficients are

$$a_0 = \frac{|e^{2i\theta} - \mu(re^{i\theta})|^2}{1 - |\mu(re^{i\theta})|^2}, \qquad a_1 = \frac{|e^{2i\theta} + \mu(re^{i\theta})|^2}{1 - |\mu(re^{i\theta})|^2},$$

$\mu = \overline{\partial} f / \partial f$ is the complex dilatation. e.g. f conformal, $\mu \equiv 0$.

Is g computable for conformal maps?

Yes. . . in fact

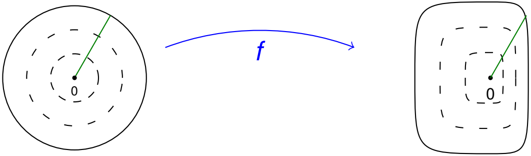
$$\min\left\{g(f) : f \text{ maps } \mathbb{D} \text{ to } \Omega \text{ conformally}\right\}$$
$$= \frac{1}{L}\left\{\left(\frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta})|^2\, d\theta\right)^2 - \left|\frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta})|^2 e^{i\theta}\, d\theta\right|^2\right\}^{1/4}$$

Drawback: if $\partial\Omega$ has corners then $|f'|^2$ can blow up.
So this method works best for smooth domains.

Is g computable for starlike domains?

Yes...

Starlike domain $\Omega = \{re^{i\theta} : \theta \in [0, 2\pi], 0 \leq r < R(\theta)\}$



for *radius function* function $R(\theta)$. Then $\boxed{g = \sqrt{g_0 g_1}}$ where

$$g_0 = 1 + \frac{1}{2\pi} \int_0^{2\pi} (\log R)'(\theta)^2 \, d\theta,$$

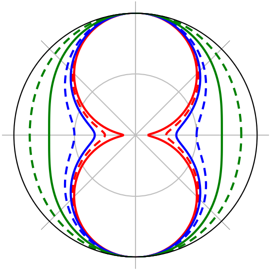
$$g_1 = \frac{1}{2\pi L^2} \int_0^{2\pi} \left(R(\theta)^2 + R'(\theta)^2\right) d\theta \,,$$

Conclusion.

If we know the radius function for Ω, then g is computable.

Optimal choice of origin is center of symmetry, if it exists.

Example — hippopedes (inverted ellipses)



$1-\varepsilon^2$	1/16	1/9	1/4	1/2	3/4	1
numerical g	1.1016	1.0924	1.0692	1.0281	1.0056	1
conformal g	1.6078	1.4302	1.2112	1.0627	1.0115	1
starlike g	1.4909	1.3214	1.1378	1.0366	1.0064	1

Conformal and starlike methods both give better results for eigenvalue sums than Hersch–Payne–Schiffer, for not-too-eccentric hippopedes, since $g < 3/2$ in those cases.

Future directions

1. What kind of quasiconformal maps have purely angular dilatation? Do they give computable geometric factor g?

2. Hersch–Payne–Schiffer showed how to maximize individual Steklov eigenvalues (union of disks).

What domains maximize *combinations* of eigenvalues?

e.g. Disk is known to maximize geometric mean $\sqrt{\sigma_1 \sigma_2}\, L$.

What about the arithmetic mean $\frac{1}{2}(\sigma_1 + \sigma_2)L$?
Numerics suggest a stadium-like domain... open problem.

From sums to heat trace by majorization (Hardy, Littlewood, Pólya)

If $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \leq b_2 \leq b_3 \leq \cdots$ and

$$a_1 + \cdots + a_n \leq b_1 + \cdots + b_n \qquad \forall n \geq 1$$

then

$$\Phi(a_1) + \cdots + \Phi(a_n) \leq \Phi(b_1) + \cdots + \Phi(b_n) \qquad \forall n \geq 1$$

for all concave increasing functions Φ.
(*Fun exercise.* Prove it for $n = 1, 2$.)

Example:
using $\Phi(c) = -\exp(-ct)$ gives heat trace