Nearly radial modes for Neumann Laplacian on highly symmetric domains.

Joint work with N. Nigam and B. Young

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Neumann Laplacian

$$\Delta u = \mu u \qquad \text{in } \Omega$$
$$\partial_n u = 0 \qquad \text{on } \partial\Omega.$$

Then

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \to \infty.$$

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Theorem (Hoffmann-Ostenhof, '13)

If Ω is a rectangle, and u is a Neumann eigenfunction with $u > 0$ on $\partial\Omega$ (we may have a mix of u_n!), then u is constant inside Ω.

Extensions to highly symmetric domains (Nigam-S.-Young, '15)

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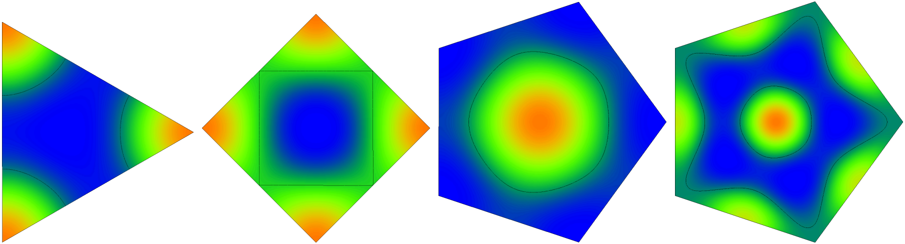

Explicit eigenvalues on $[-\pi, \pi]^3$:

Multiply indexed sequence of integers:

$$\mu_{i,j,k} = i^2 + j^2 + k^2, \qquad i, j, k \geq 0,$$

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μ is a square.

A restriction of u to a face of the cube gives a mix of Neumann eigenfunctions of the face. And these integrate to 0, unless constant eigenfunction involved. Hence

$$\mu_{i,j,k} = m^2.$$

Number theory in $d = 2, 3, 4$ only.

Parity of the integers on the right of

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Dimension 6:

Arbitrary integers may be involved:

$$6^2 = 2 \times 4^2 + 4 \times 1^2 = 5^2 + 2 \times 2^2 + 3 \times 1^2.$$

No common periods, no apparent 0 point.

Lattice points on a face:

For $k = 0, \cdots, m-1$ let $x_i^{(k)} = \pi\left(1 - \frac{2k-1}{m}\right)$.
On the face $x_n = \pi$ build a lattice $\mathcal{L}$ from points with coordinates $x_i^{(k)}$.

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Small cubes.

Pick a so that

$$am = (d-1)\sin(am) \qquad (a = 0 \text{ if } d = 2!).$$

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Integration over the union of cubes $\mathcal{C}$ gives 0!

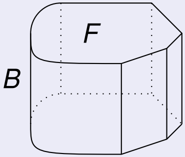
$$\int_{\mathcal{C}} u = 0 \qquad \Longrightarrow \qquad \exists_c \int_c u \geq 0.$$

But u cannot satisfy Dirichlet and Neumann condition on c, hence $u > 0$ somewhere. Similarly $u < 0$ somewhere.

Sloshing problem

Let W be a container, with sides B and free fluid surface F. Then

$$\begin{aligned} \Delta u &= 0 && \text{in } W, \\ \partial_n u &= \lambda u && \text{on } F, \\ \partial_n u &= 0 && \text{on } B, \\ \Big(\int_F u &= 0. \Big) \end{aligned}$$



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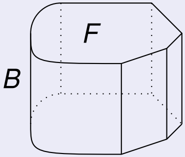

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Cylindrical containers

If W is a cylinder with cross-section F and height l, then

$$\lambda_n(W) = \sqrt{\mu_n(F)} \tanh\left(\sqrt{\mu_n(F)} l\right).$$

And eigenfunctions are also related.

Hot-spots conjecture (Neumann problem)

Any/all eigenfunctions for the eigenvalue μ_2 have their global max/min on the boundary of Ω.

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Partial results:

- Burdzy, Bañuelos (line of symmetry and further restrictions)
- Jerison, Nadirashvili (two lines of symmetry)
- Atar, Burdzy (convexity and Lipschitz boundary with constant ≤ 1)
- Burdzy, Werner (counterexample with 2 holes)
- Polymath7: Acute triangles still not solved (some cases: S. ’15).

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High-spots conjecture (sloshing problem)

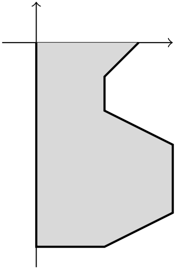
Direct translation: Highest spot of an eigenfunction for λ_2 is on the boundary of the container.
Cylindrical case reduces to hot-spots problem.

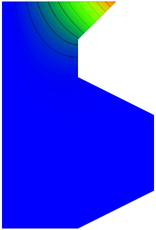
Theorem (Kulczycki, Kwaśnicki, '12)

If W is axisymmetric (F is a disk), then the high spot is at:

- *The boundary if sides meet F at an acute or right angle.*

Profiles (rotate to get an axisymmetric container):





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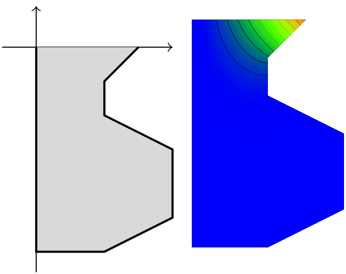

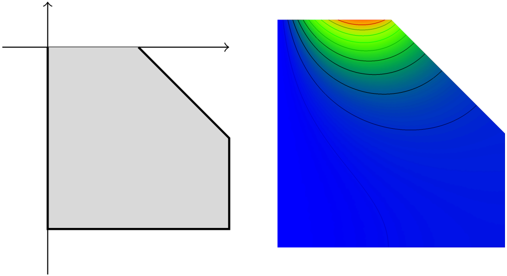

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Caveat: Fundamental mode might be radial, high-spot inside.

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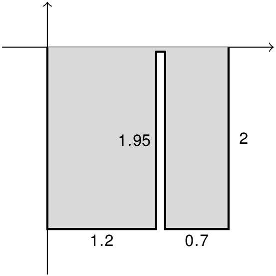
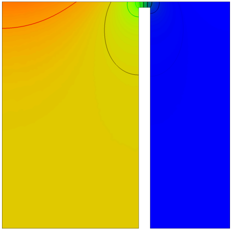




Sloshing in a cup (nearly radial modes).

It is possible to slosh a fluid so that the surface moves in unison on the boundary of a round cup.

Nigam-S.-Young

The same in a cup shaped like regular polygon with $n > 4$ sides (though with some height variations).

Hoffmann-Ostenhof

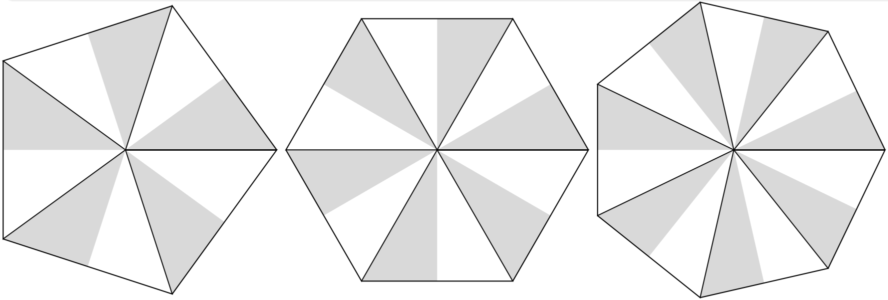
In a square cup one can slosh so that four or more points do not move, and the rest of the boundary values have the same sign.

Nigam-S.-Young

In a triangular cup water always moves up in one place, down in another (on the boundary).

Simplifications for regular polygons with $n \geq 5$.

It is enough to study an appropriape right triangle.



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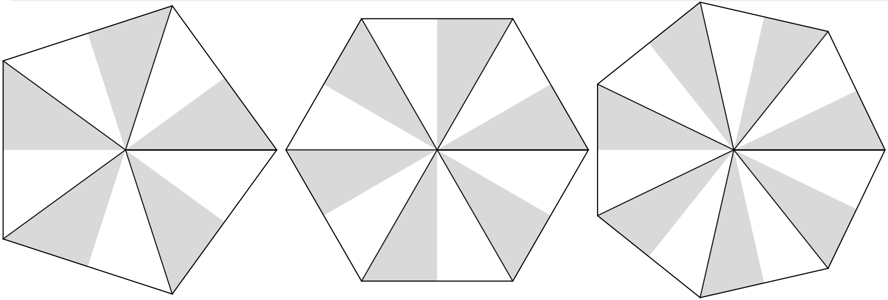

We want the nodal line to avoid the shortest side of the right triangle.

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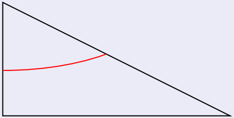
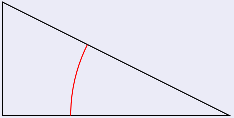
Miyamoto, S.

The eigenfunction for μ_2 is monotonic in the directions of the perpendicular sides, so the nodal line is a graph of an increasing function.

Two cases left:



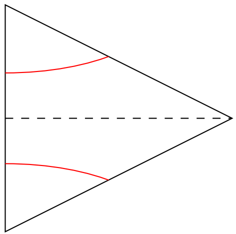


6 or more sides.

The right triangle has an angle $\leq \pi/6$.

Laugesen-S. '10

Isosceles triangle built using two such triangles has symmetric eigenfunction for μ_2.



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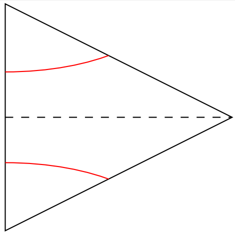

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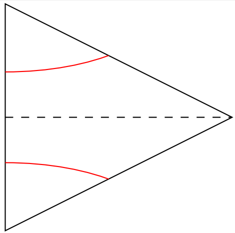
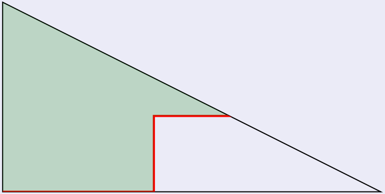
Courant nodal domain theorem implies that Case 1 (shown) is impossible, due to too many nodal domains.



Regular pentagon: eigenfunction antisymmetric on associated isosceles triangle! Assume Case 1 holds.

Domain monotonicity (special case of Harrell '06)

Put Dirichlet condition on red boundary, Neumann otherwise on green subdomain of the triangle.



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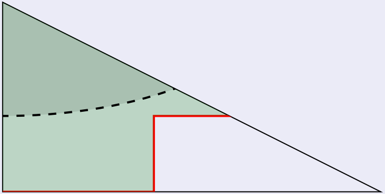

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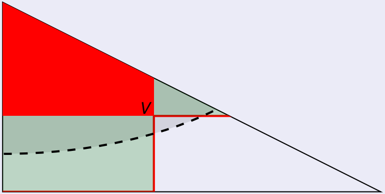
The nodal line (dashed) cannot be inside of the green domain, since dark green domain is smaller.

Conclusion:



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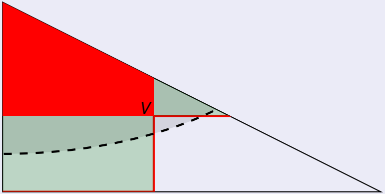

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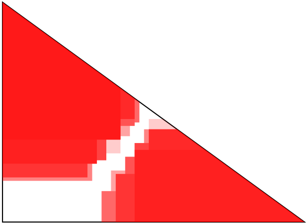

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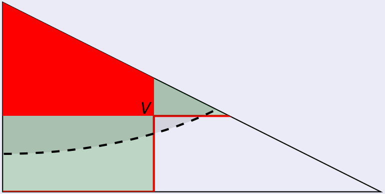

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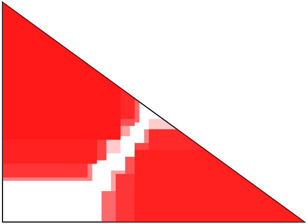

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Finally:

The nodal line cannot touch red subsets. Yet the complement of the upper one has too large eigenvalue, violating domain monotonicity.

Nonconforming CR-FEM: lower bound (Carstensen-Gedicke, '14)

Lower bound for the smallest eigenvalue of a FEM matrix, gives a lower bound for mixed Dirichlet-Neumann eigenvalue (correction depends on the mesh resolution).

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Find LU or LDL' decomposition for the matrix and check diagonal entries in U or D (we did this using interval arithmetic and exact rational representations).