The Brezis–Nirenberg problem on $S^n$, in spaces of fractional dimension.

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**Motivation (The Lane–Emden equation):**

The equation

\[-\Delta u = u^p\] (1)

for \(u > 0\) in a ball of radius \(R\) in \(\mathbb{R}^3\), with Dirichlet boundary conditions, is called, in physics, the Lane–Emden equation of index \(p\). It was introduced in 1869 by Homer Lane, who was interested in computing both the temperature and the density of mass on the surface of the Sun. Unfortunately Stefan’s law was unknown at the time (Stefan published his law in 1879). Instead, Lane used some experimental results of Dulong and Petit and Hopkins on the rate of emission of radiant energy by a heated surface, and he got the value of 30,000 degrees Kelvin for the temperature of the Sun, which is too big by a factor of 5. Then he used his value of the temperature together with the solution of (1) with \(p = 3/2\), to estimate the density \(u\) near the surface.
Motivation (The Lane–Emden equation):

After the Lane–Emden equation was introduced, it was soon realized that it only had bounded solutions vanishing at \( R \) if the exponent is below 5. In fact, for \( 1 \leq p < 5 \) there are bounded solutions, which are decreasing with the distance from the center. In 1883, Sir Arthur Schuster constructed a bounded solution of the Lane–Emden equation in the whole \( \mathbb{R}^3 \) vanishing at infinity. This equation on the whole \( \mathbb{R}^3 \), with exponent \( p = 5 \) plays a major role in mathematics. It is the Euler–Lagrange equation equation that one obtains when minimizing the quotient

\[
\frac{\int (\nabla u)^2 \, dx}{\left( \int u^6 \, dx \right)^{1/3}}. \tag{1}
\]

This quotient is minimized if \( u(x) = 1/(|x|^2 + m^2)^{1/2} \). The minimizer is unique modulo multiplications by a constant, and translations. This function \( u(x) \), is precisely the function determined by A. Schuster, up to a multiplicative constant. Inserting this function \( u \) back in (1), gives the classical Sobolev inequality (S. Sobolev 1938),

\[
\frac{\int (\nabla u)^2 \, dx}{\left( \int u^6 \, dx \right)^{1/3}} \geq 3\left( \frac{\pi}{2} \right)^{4/3}, \tag{2}
\]

for all functions in \( \mathcal{D}^1(\mathbb{R}^3) \).
The Brezis–Nirenberg problem on $\mathbb{R}^N$

In 1983 Brezis and Nirenberg considered the nonlinear eigenvalue problem,

$$-\Delta u = \lambda u + |u|^{4/(n-2)}u,$$

with $u \in H^1_0(\Omega)$, where $\Omega$ is bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$. Among other results, they proved that if $n \geq 4$, there is a positive solution of this problem for all $\lambda \in (0, \lambda_1)$ where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $\Omega$. They also proved that if $n = 3$, there is a $\mu(\Omega) > 0$ such that for any $\lambda \in (\mu, \lambda_1)$, the nonlinear eigenvalue problem has a positive solution. Moreover, if $\Omega$ is a ball, $\mu = \lambda_1/4$. 
The Brezis–Nirenberg problem on $\mathbb{R}^N$

For positive radial solutions of this problem in a (unit) ball, one is led to an ODE that still makes sense when $n$ is a real number rather than a natural number.

Precisely this problem with $2 \leq n \leq 4$, was considered by E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations, **156** (1999), pp. 407–426.

Among other things Jannelli proved that this problem has a positive solution if and only if $\lambda$ is such that

$$j_{-(n-2)/2,1} < \sqrt{\lambda} < j_{+(n-2)/2,1},$$

where $j_{\nu,k}$ denotes the $k$–th positive zero of the Bessel function $J_{\nu}$. 


The Brezis–Nirenberg problem on $\mathbb{R}^N$

\[ j_{0,1} = 2.4048 \ldots \]

\[ \pi / 2 \]
The Brezis–Nirenberg problem on $\mathbb{S}^N$

We consider the nonlinear eigenvalue problem,

$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H^1_0(\Omega)$, where $\Omega$ is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. In dimension 3, Bandle and Benguria (JDE, 2002) proved that this problem has a unique positive solution if and only if

$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2}$$

where $\theta_1$ is the geodesic radius of the ball.
The Brezis–Nirenberg problem on $S^N$

The Brezis–Nirenberg problem on $S^3$

“Laplacians and Heat Kernels: Theory and Applications”

Brezis-Nirenberg Problem on $S^N$, $2 < N < 4$. 
The Brezis–Nirenberg problem on $\mathbb{S}^N$

For positive radial solutions of this problem one is led to an ODE that still makes sense when $n$ is a real number rather than a natural number.

In this talk I will consider precisely that problem with $2 < n < 4$. Our main result is that in this case one has a positive solution if and only if $\lambda$ is such that

$$\frac{1}{4}[(2\ell_2 + 1)^2 - (n - 1)^2] < \lambda < \frac{1}{4}[(2\ell_1 + 1)^2 - (n - 1)^2]$$

where $\ell_1$ (respectively $\ell_2$) is the first positive value of $\ell$ for which the associated Legendre function $P_{\ell}^{(2-n)/2}(\cos \theta_1)$ (respectively $P_{\ell}^{(n-2)/2}(\cos \theta_1)$) vanishes.
The Brezis–Nirenberg problem on $S^N$

Strategy of the Proof:

For the nonexistence of solutions:

i) Use a Rellich–Pohozaev’s type argument for values of $\lambda$ below the lower bound.

ii) Multiply the ODE by the first eigenfunction of the Dirichlet problem to rule out the values of $\lambda$ larger than the upper bound.

For the Existence part, use a variational characterization of $\lambda$ and a Brezis–Lieb lemma (or, alternatively, a concentration compactness argument).
Equation for the first Dirichlet Eigenvalue of a geodesic cap:

The equation that determines the first Dirichlet eigenvalue is given by,

\[ u''(\theta) + (n - 1) \frac{\cos \theta}{\sin \theta} u'(\theta) + \lambda u = 0, \quad (1) \]

with \( u(\theta_1) = 0 \), and \( u(\theta) > 0 \) in \( 0 \leq \theta < \theta_1 \) (here \( \theta_1 \) is the radius of the geodesic ball in \( S^n \), and \( 0 < \theta_1 \leq \pi \)). For geodesic balls contained in a hemisphere, \( 0 < \theta_1 \leq \pi/2 \).

Let \( \alpha = -(n - 2)/2 \), and set

\[ u(\theta) = (\sin \theta)^\alpha v(\theta). \quad (2) \]

Then \( v(\theta) \) satisfies the equation,

\[ v''(\theta) + \frac{\cos \theta}{\sin \theta} v'(\theta) + \left( \lambda + \alpha(\alpha - 1) - \frac{\alpha^2}{\sin^2 \theta} \right) v = 0. \quad (3) \]
Equation for the first Dirichlet Eigenvalue of a geodesic cap:

In the particular case when \( n = 3, \alpha = -1/2 \) and this equation becomes,

\[
v''(\theta) + \frac{\cos \theta}{\sin \theta} v'(\theta) + \left( \lambda + \frac{3}{4} - \frac{1}{4 \sin^2 \theta} \right) v = 0. \tag{1}\]

whose positive regular solution is given by,

\[
v(\theta) = C \frac{\sin \left( \sqrt{1 + \lambda} \theta \right)}{\sqrt{\sin \theta}} \tag{2}\]

hence, in this case,

\[
u(\theta) = C \frac{\sin \left( \sqrt{1 + \lambda} \theta \right)}{\sin \theta}. \tag{3}\]

Imposing the boundary condition \( u(\theta_1) = 0 \), in the case \( n = 3 \), we find that,

\[
\lambda_1(\theta_1) = \frac{\pi^2}{\theta_1^2} - \theta_1^2. \tag{4}\]
Equation for the first Dirichlet Eigenvalue of a geodesic cap:

The regular solution of the ODE for the first Dirichlet eigenvalue (for general $n$) is given by

$$v(\theta) = \ell^m(\cos \theta),$$

(1)

where $P^m_\ell(x)$ is an associated Legendre function, with indices,

$$m = \alpha = (2 - n)/2,$$

(2)

and

$$\ell = \frac{1}{2} \left( \sqrt{1 + 4\lambda - 4\alpha + 4\alpha^2} - 1 \right).$$

(3)
Existence of solutions (stereographic projection):

\[ x = \tan\left(\frac{\theta}{2}\right) \]
\[ \theta = 2 \arctan(x) \]
\[ d\theta = \frac{2}{1 + |x|^2} \, dx \]
\[ d\theta = q(x) \, dx \]

“Laplacians and Heat Kernels: Theory and Applications”
Brezis-Nirenberg Problem on \( S^N, \, 2 < N < 4 \).
Existence of solutions:

Let $D$ be a geodesic ball on $\mathbb{S}^n$. The solutions of

$$\begin{cases}
-\Delta_{\mathbb{S}^n} u = \lambda u + u^p & \text{on } D \\
u > 0 & \text{on } D \\
u = 0 & \text{on } \partial D,
\end{cases}$$

where $p = \frac{n+2}{n-2}$ correspond to minimizers of

$$Q_{\lambda}(u) = \frac{\int (\nabla u)^2 q^{n-2} \, dx - \lambda \int u^2 q^n \, dx}{\left(\int u^{2n} q^n \, dx\right)^{\frac{n-2}{n}}}.$$

(1)

Here $q(x) = \frac{2}{1+|x|^2}$, so that the line element of $S^n$ is proportional to the line element of the Euclidean space, i.e., $ds = q(x)dx$ through the standard stereographic projection.
**Existence of solutions:**

In 1999 Bandle and Peletier (Math. Annalen) proved that for domains contained in the hemisphere the infimum of the Rayleigh quotient of the Sobolev inequality on $S^n$ is not attained, and the value of the sharp constant is precisely the same as in the Euclidean Space of the same dimension.

Thus, one can use the Brezis–Lieb classical lemma (1983) or alternatively a concentration compactness argument to show that if there is a function on the right space that satisfies $Q(\lambda)(u) < S$, then the minimizer for $Q_\lambda$ is attained. The minimiser is positive and satisfies the Brezis–Nirenberg equation.

To construct the desired function we use the Schuster function centred at the North Pole, multiplied by a cutoff function introduced to satisfy the Dirichlet boundary condition.
Existence of solutions:

Let $\varphi$ be a smooth function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(1) = 0$. For $\epsilon > 0$, let

$$u_\epsilon(r) = \frac{\varphi(r)}{(\epsilon + r^2)^{\frac{n-2}{2}}}.$$  \hspace{1cm} (1)

We claim that for $\epsilon$ small enough, $Q_\lambda(u_\epsilon) \leq S$. In the next three claims we compute $||\nabla u_\epsilon||^2_2$, $||u_\epsilon||^2_{p+1}$ and $||u_\epsilon||^2_2$.

$$\int (\nabla u_\epsilon)^2 q^{n-2} \, dx = \omega_n \int_0^R \varphi'(r)^2 r^{3-n} q^{n-2} \, dr - \omega_n (n-2)^2 \int_0^R \varphi(r)^2 r^{3-n} q^{n-1} \, dr$$
$$+ \omega_n n(n-2) 2^{n-2} D_n \epsilon^{\frac{2-n}{2}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}),$$  \hspace{1cm} (2)

where

$$D_n = \frac{1}{2} \frac{\Gamma \left( \frac{n}{2} \right)^2}{\Gamma(n)}$$
$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} \right)}.$$  \hspace{1cm} (3)
Existence of solutions:

\[
\int u^2 q^n \, dx = \omega_n \int_0^R q^n r^{3-n} \varphi^2 \, dr + O(\epsilon^{\frac{4-n}{2}}).
\]

\[
\left(\int u^{\frac{2n}{n-2}} q^n \, dx\right)^{\frac{n-2}{n}} = \omega_n \, 2^{n-2} \epsilon^{\frac{2-n}{2}} D_n^{\frac{n-2}{n}} + O(\epsilon^{\frac{4-n}{2}}),
\]

where

\[
D_n = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}.
\]
Existence of solutions:

\[ Q_\lambda(u_\varepsilon) = n(n - 2)(\omega_n D_n)^{\frac{2}{n}} + \varepsilon^{\frac{n-2}{2}} C_n \left[ \int_0^R r^{3-n} \left( q^{n-2} \varphi'^2 - (n-2)q^{n-1} \varphi^2 - \lambda q^n \varphi^2 \right) \, dr \right] + \mathcal{O}(\varepsilon), \]

(1)

where \( C_n = \omega_n n^{2-n} D_n^{\frac{2-n}{n}} \).

Notice that

\[ n(n - 2)(\omega_n D_n)^{\frac{2}{n}} = \pi n(n - 2) \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma(n)} \right)^{\frac{2}{n}}, \]

which is precisely the Sobolev critical constant \( S \).
Existence of solutions:

Let

\[ T(\varphi) = \int_0^R r^{3-n} \left( q^{n-2} \varphi'^2 - (n - 2)^2 q^{n-1} \varphi^2 - \lambda q^n \varphi^2 \right) \, dr. \]

It suffices to show that \( T(\varphi) \) is positive. The associated Euler equation is

\[ \varphi''(r) + (3 - n) \frac{\varphi'(r)}{r} + (n - 2) \frac{\varphi'(r)q'(r)}{q(r)} + (n - 2)^2 q(r)\varphi(r) + \lambda q(r)^2 \varphi(r) = 0. \]

Setting \( r = \tan \theta/2 \), and

\[ \varphi = \sin^b \frac{\theta}{2} \sin^a \theta \varphi, \]

where \( b = 2n - 4 \) and \( a = \frac{1}{2}(6 - 3n) \), and multiplying the equation through by \( \sin^{-b} \frac{\theta}{2} \sin^{-a} \theta \) we obtain

\[ \ddot{\varphi}(\theta) + \cot \theta \dot{\varphi}(\theta) + \left( \lambda + \frac{n(n - 2)}{4} - \frac{(n - 2)^2}{4 \sin^2 \theta} \right) \varphi = 0. \]
Existence of Positive Solutions:

From here it follows that $T(\varphi) < 0$ provided

$$\lambda > \frac{1}{4}[(2\ell_2 + 1)^2 - (n - 1)^2],$$

where $\ell_2$ is the first positive value of $\ell$ for which the associated Legendre function $P_{\ell}^{(2-n)/2}(\cos \theta_1)$ vanishes.

This concludes the proof of the existence of positive solutions.
Nonexistence of solutions (a Rellich–Pohozaev’s argument):

For radial solutions, the original nonlinear eigenvalue problem,

$$-\Delta_{S^n} u = \lambda u + u^p$$  \hspace{1cm} (1)

where $u > 0$ on $D$, and $u = 0$ on $\partial D$ can be written as

$$-(\sin^{n-1} \theta \, u')' = u^p + \lambda u,$$  \hspace{1cm} (2)

with initial conditions $u'(0) = 0$, and $u(\theta_1) = 0$. 
Here $D$ denotes a geodesic cap of geodesic radius $\theta_1$, and $'$ denotes derivative with respect to $\theta$. 
Nonexistence of solutions (a Rellich–Pohozaev’s argument):

Multiplying equation (1) by \( g(\theta)u'(\theta)\sin^{2n-2}\theta \) we obtain

\[
- \int_{0}^{\theta_1} (\sin^{n-1}\theta u')'u'g \sin^{n-1}\theta \, d\theta = \int_{0}^{\theta_1} \left( \frac{u^{p+1}}{p+1} \right)'g \sin^{2n-2}\theta \, d\theta + \lambda \int_{0}^{\theta_1} \left( \frac{u^2}{2} \right)'g \sin^{2n-2}\theta \, d\theta
\]

Integrating by parts we have that

\[
\int_{0}^{\theta_1} u^2 \left( \frac{g'}{2} \sin^{2n-2}\theta \right) \, d\theta + \int_{0}^{\theta_1} \frac{u^{p+1}}{p+1} \left( g' \sin^{2n-2}\theta + g(2n-2) \sin^{2n-3}\theta \cos\theta \right) \, d\theta
\]

\[
+ \lambda \int_{0}^{\theta_1} \frac{u^2}{2} \left( g' \sin^{2n-2}\theta + g(2n-2) \sin^{2n-3}\theta \cos\theta \right) \, d\theta = \frac{1}{2} \sin^{2n-2}\theta_1 u'(\theta_1)^2 g(\theta_1).
\]

(1)
Nonexistence of solutions (a Rellich–Pohozaev’s argument):

On the other hand, setting \( h = \frac{1}{2} g' \sin^{n-1} \theta \) and multiplying equation (1) by \( h(\theta) u(\theta) \sin^{n-1}(\theta) \) we obtain

\[
- \int_0^\theta (\sin^{n-1} \theta u')' h u \, d\theta = \int_0^\theta h u^{p+1} \sin^{n-1} \theta \, d\theta + \lambda \int_0^\theta h u^2 \sin^{n-1} \theta \, d\theta.
\]

Integrating by parts we obtain

\[
\int_0^\theta u'^2 h \sin^{n-1} \theta \, d\theta = \int_0^\theta u^{p+1} h \sin^{n-1} \theta \, d\theta
\]

\[
+ \int_0^\theta u^2 \left( \lambda h \sin^{n-1} \theta + \frac{1}{2} h'' \sin^{n-1} \theta + \frac{1}{2} h'(n-1) \sin^{n-2} \theta \cos \theta \right) \, d\theta.
\]
Nonexistence of solutions (a Rellich–Pohozaev’s argument):

\[
\frac{1}{2} \sin^{2n-2} \theta_1 u' (\theta_1)^2 g(\theta_1) = \int_{0}^{\theta_1} B u^{p+1} d\theta + \int_{0}^{\theta_1} A u^2 d\theta, \quad (1)
\]

by hypothesis \(g(\theta_1) \geq 0\), it follows that the left hand side is nonnegative. We will show that there exist a choice of \(g\) so that for appropriate values of \(\lambda\), \(A \equiv 0\), and \(B\) is negative, thus obtaining a contradiction.

Here,

\[
A = \sin^{2n-2} \theta \left[ \frac{g'''}{4} + \frac{3}{4} g''(n - 1) \cot \theta 
+ g' \left( \frac{(n - 1)(n - 2) \cot^2 \theta}{4} - \frac{n - 1}{4} + \lambda \right)
+ \lambda g(n - 1) \cot \theta \right],
\]

and

\[
B \equiv \frac{1}{2} g' \sin^{2n-2} \theta + \frac{g' \sin^{2n-2} \theta}{p + 1} + \frac{(2n - 2) g \sin^{2n-3} \theta \cos \theta}{p + 1}. \quad (2)
\]
Nonexistence of solutions (a Rellich–Pohozaev’s argument):

Setting \( f = g \sin^2 \theta \) and writing \( m = n - 3 \), the equation \( A = 0 \) is equivalent to,

\[
\sin^{2m+2} \theta \left[ \frac{f'''}{4} + \frac{3}{4} m \cot \theta f'' + f' \left( \frac{m(2m-5)}{4} \cot^2 \theta + \frac{4-m}{4} + \lambda \right) \right. \\
+ f \left( m(1-m) \cot^3 \theta + 2m \cot \theta + \lambda m \cot \theta \right] = 0
\]

(1)

An appropriate solution is given by,

\[
f(\theta) = \sin^{1-m} \theta P_\ell^\nu (\cos \theta) P_\ell^{-\nu} (\cos \theta),
\]

where \( \nu = \frac{m+1}{2} \) and \( \ell = \frac{1}{2} \left( \sqrt{4\lambda + (m+2)^2} - 1 \right) \).

Using the raising and lowering relations for the Associated Legendre functions and some work!, one can show that \( B < 0 \) for this choice of \( f \), provided

\[
\lambda < \frac{1}{4} [(2\ell_2 + 1)^2 - (n-1)^2],
\]

where \( \ell_2 \) is the first positive value of \( \ell \) for which the associated Legendre function \( P_{\ell}(2-n)/2 (\cos \theta_1) \) vanishes.
The Brezis–Nirenberg problem on $S^N$

Work in Progress:

i) The Brezis–Nirenberg problem on $\mathbb{H}^n$ with fractional $n$. (Soledad Benguria).

ii) In $\mathbb{S}^3$ Wei, Bandle, Peletier and Brezis studied and classified all possible solutions of the BN problem for geodesic balls beyond the hemisphere. We are investigating the behaviour of these families of solutions in the case of fractional $2 < n < 4$. 