Trees and the Agmon Metric

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How do L^2 eigenfunctions decay on a quantum graph?

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Prior work:

- Liouville and Green for 1D, Agmon: negative energies
- Hislop-Post: Worked with "radial trees," i.e. identical lengths and identical branching numbers, and a potential added at the vertices as a vertex condition; positive energy localization
- Random lengths model

Agmon philosophy

For negative eigenvalues (actually $E < \liminf V$), use tricky integration by parts to show that $\psi \in Ł^2$ and $H\psi = E\psi \implies exp(\rho(0,x))\psi \in L^2$, where ρ is a metric. Once L^2 exponential decay is established, they can be used to establish pointwise estimates showing exponential decrease.
The method works essentially if $x : V(x) - E \leq 0$ is compact, and the expected metric is essentially an "action integral" of $(V(x) - E)_+^{1/2}$ over the minimizing path.

Notations:

Our setup:

- Quantum graph Γ; edges are segments of the real line
- A function space $\mathcal{K}$ on Γ so that for $\phi \in \mathcal{K}$
  1. ϕ is twice differentiable on the edges
  2. ϕ is continuous at the vertices

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  1. ϕ is twice differentiable on the edges
  2. ϕ is continuous at the vertices
  3. if v is a vertex, e_j are the edges adjacent to v, and ϕ_{e_j} is ϕ restricted to the edge e_j, then $\sum \phi'_{e_j}(v) = 0$ where the derivative is taken in the direction away from the vertex v
     1. whatever comes in must come out
     2. Kirchoff condition
  4. Notice: if $f, g \in \mathcal{K}$ then so is fg.
- Differential operator $H = -\Delta + V(x)$ acting on $\mathcal{K}$

Case study: the Line

Definition 1

Let $\rho_E(x,y) = \int_x^y \sqrt{(V(t)-E)_+}dt$.

Notes:

- In multiple dimensions or on graphs with non-trivial topology, take the minimum over all possible paths from x to y.

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Theorem 2 (Agmon)

Let $H = -\Delta + V$ with V real and continuous be a closed operator bounded below with $\sigma(H) \subset \mathbb{R}$. Suppose E is an eigenvalue of H and that $\mathrm{supp}(E - V(x))_+$ is compact. Suppose $\psi \in L^2$ is an eigenfunction of H. Then for any $\epsilon > 0$ there exists a constant c_ϵ such that

$$\int e^{2(1-\epsilon)\rho_E(x)}|\psi(x)|^2 dx \leq c_\epsilon$$

adapted from Hislop-Sigal book

Agmon on the Line: Proof ingredient # 1

Lemma 3

For $\phi \in L^2$ *and* F_α *bounded and satisfying* $V - E - \left|\frac{F'_\alpha}{F_\alpha}\right|^2 > \delta$, *we get that*

$$\langle F_\alpha \phi, (H - E)\frac{1}{F_\alpha}\phi\rangle \geq \delta \|\phi\|^2$$

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Proof:

$$\left\langle F_\alpha \phi, (H - E) \frac{1}{F_\alpha} \phi \right\rangle = \int (V - E)|\phi|^2 + (F_\alpha \phi)' \left(\frac{\phi}{F_\alpha} \right)' + BT$$

$$= \int (V - E)|\phi|^2 + |\phi'|^2 - \left| \frac{F'_\alpha \phi}{F_\alpha} \right|^2 \geq \delta \|\phi\|^2.$$

Dropped the term $|\phi'|^2$, since positive. Integrated by parts, since $\phi \in L^2$ and F_α is bounded. Note δ is independent of the upper bound on F_α.

Proof Ingredient # 2

Lemma 4

Let η *be a smoothed characteristic function of* $\{V - E > \delta\}$ *such that* η' *has compact support and* ψ *be an* L^2 *solution of* $(H - E)\psi = 0$. *Then*

$$\langle F_\alpha^2 \eta\psi, (H - E)\eta\psi \rangle = \left\langle \eta' \left(\eta F_\alpha^2\right)' \psi, \psi \right\rangle \leq C\|\psi\|^2. \tag{3.1}$$

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- Here we have taken ϕ from before to be $\eta F_\alpha \psi$, so that $\delta\|\eta F_\alpha \psi\|^2 \leq C\|\psi\|^2$.

Proof ingredient # 3

We want to take F as large as possible, but still satisfying the constraint $V - E - \left|\frac{F'}{F}\right|^2 > \delta$, so we construct it by

$$F(x) = e^{(1-\delta)\int_0^x (V(t)-E)_+} = e^{(1-\delta)\rho_E(x)}$$

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We observe that

- (i) $F_\alpha(x) < F(x)$,
- (ii) $|F'_\alpha(x)| < |F'(x)|$,
- (iii) $\left|\frac{F'_\alpha(x)}{F_\alpha(x)}\right| < \left|\frac{F'(x)}{F(x)}\right|$.

Then can take $\alpha \to 0$, since the constants throughout do not depend on α.

For a quantum tree:

- Same argument carries over to rooted trees with few modifications
- Integration by parts still works because of the Kirchoff condition
- Can show that if $\psi \in L^2(\Gamma)$ then $e^{(1-\epsilon)\rho_E(x)}\psi \in L^2$, where $\rho = \int_0^x (V(t) - E)_+^{1/2}$ and the integral is taken over the unique path from the root to x
- By a standard method can go from L^2 estimates to pointwise.
- Are we done?

Case study: the regular tree

- Regular tree: starts at a root, each edge has length L and at each vertex splits into b edges, e.g. the Bethe Lattice.
- Fix $E < 0$. Consider $H = -\Delta$, i.e. $V = 0$ and look for a L^2 solution.

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- Fix $E < 0$. Consider $H = -\Delta$, i.e. $V = 0$ and look for a L^2 solution.
- The problem: Does there exist an L^2 solution and does it decay faster than $e^{-\sqrt{|E|}x}$, which would be the corresponding decay on the line.

Construction for the Regular Tree

Transfer matrix: $T = \begin{pmatrix} \cosh kL & \sinh kL \\ \frac{1}{b}\sinh kL & \frac{1}{b}\cosh kL \end{pmatrix}$

- Both eigenvalues λ_1, λ_2 are real Since $\det T = 1/b$ and $\mathrm{Tr}\, T = \left(1 + \frac{1}{b}\right) \cosh kL > 2/\sqrt{b}$
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The eigenvalue λ_1 is given by

$$\begin{aligned} \lambda_1 &= \left(\frac{1}{2} + \frac{1}{2b}\right)\cosh kL - \sqrt{\left(\left(\frac{1}{2} + \frac{1}{2b}\right)\cosh kL\right)^2 - 1/b} \\ &= \frac{1}{\left(\frac{b}{2} + \frac{1}{2}\right)\cosh kL + \sqrt{\left(\left(\frac{b}{2} + \frac{1}{2}\right)\cosh kL\right)^2 - b}} \le \frac{1}{b\cosh kL} \end{aligned} \tag{3.2}$$

- the above solution is in L^2 for the tree since

$$\int_\Gamma |\phi|^2 = C \sum_n b^n \lambda_1^{2n}.$$

If $\lambda_1 < \alpha/\sqrt{b}$ for $\alpha < 1$ then the above sum converges.

- The factor of $\frac{1}{b^n}$ makes the pointwise decay faster than the case of the line, where the decay is just e^{-kx}, but this comparison is "unfair."

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- If a solution is to be in L^2 for a tree the $1/\sqrt{b}$ factor is required for convergence. However, we have a factor of $1/b$ instead, which means that even if we consider partial integrals, the decay on the tree will be faster than on the line.

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- Does this hold more generally?

The Agmon Metric for a General Tree

We label the maximum out-degree on the tree to be d_{max} we will label each vertex by a d_{max}ary string s. We also denote the edge which terminates at vertex v_s by e_s.

Theorem 5

Suppose $\psi \in L^2$ is a solution of H. Suppose at vertex v_s, p_{sj} is the fraction of the derivative continuing down branch e_{sj}. For the path P from the root to x, let

$$\rho(x) = \int_P (V(t) - E)_+ + 2 \sum_{v \in P} \delta_V(t) \log(1/p_v).$$

Then if $\psi \in L^2$, then $e^{(1-\epsilon)\rho}\psi$ is square integrable on the path P.

Note: $e^{(1-\epsilon)\rho}\psi$ is not in $L^2(\Gamma)$ but sufficient for pointwise bounds.

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- We sacrifice continuity.
- Get boundary terms in integration by parts. Recall that $\phi = F_\alpha \eta \psi$ so for an edge e starting at v_1 and ending at v_2:

$$\int_e F\phi \frac{d^2}{dx^2}\left(\frac{1}{F}\phi\right) = \int_e \frac{d}{dx}\left(F^2\eta\psi\frac{d}{dx}(\eta\psi)\right) - \int_e \frac{d}{dx}(F\phi)\frac{d}{dx}\left(\frac{1}{F}\phi\right)$$
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- Want to take F^2 so that its discontinuity cancels with the discontinuity of $\frac{d}{dx}\psi$, so we can increase F^2 by a factor of $1/p$.
- Technicality: we have to work with F_α so the discontinuity will be in F_α then take $\lim_{\alpha\to\infty}$

Case study: the harmonic millipede

We consider a millipede with segments of length L, at each vertex v_k of which there dangles an infinitely long leg e_k. There is one leg at each vertex The energy parameter is $E = -1$, no potential on the main path, so on each bodily edge, $\psi'' = \psi$. On the dangling legs the potential is $V(x) = (x+2)^2 - 2$, where x is the distance from v_k, and we see that $V - E > 0$ on the legs. The L^2 solutions on the legs solve $\psi'' = (V(x) - (-1))\psi$, which easily gives cst.$e^{-(x+2)^2/2}$. We differentiate and find that $\psi'_{e_k}(v_k) = -2\psi(v_k)$. The transfer matrix for connecting solutions from one main-body segment to the next is

$$\begin{pmatrix} \cosh L & \sinh L \\ \sinh L - 2\cosh L & \cosh L - 2\sinh L \end{pmatrix}.$$

The eigenvalues of this are

$$\frac{e^L}{2}\left(1 \pm \sqrt{1 - \left(\frac{2}{e^L}\right)^2}\right) \approx e^{\pm L}$$

Notice:

1. Not L^2 if any $p = 0$, so $1/p$'s are ok
2. This requires a lot of information on ψ

Consider instead an averaged function Ψ.

- Can use a similar Agmon argument to show that $F\Psi \in L^2$
- Do not need information on the p's, but only get information
- Only get information on ψ in an averaged sense

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- Let $\Psi(x) = \sum_{y:\mathrm{dist}(0,y)=x} w_y \psi(x)$
- $w_y = \prod_v 1/b_v$ where the product is taken over all vertices v on the path from 0 to y and b_v is the out-degree.
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- Notice: the p's are gone!
- $\|\Psi(x)\|_{L^2}^2 \leq \int_0^\infty (\sum_y |w_y|^2)(\sum_y |\psi(y)|^2) \leq \|\psi(x)\|_{L^2}^2$
- At each vertex can increase F^2 by a factor of b_v, so $e^{(1-\epsilon)\int (V-E)_+^{1/2} + \frac{1}{2}\sum_n \log(b_n)\delta_{v_n}(t)dt}\Psi$ is also in L^2.

Questions to be settled:

1. Is the L^2 eigenfunction unique? Analogues of the the limit point - limit circle dichotomy.
2. What if we add leaves and finite subtrees to our regular infinite tree? “Obvious” that this will only improve the decay.
3. What happens in case of a more general graph? What if there are cycles?

Thank you!