Laplacians and Heat Kernels: Theory and Applications (15w5110)

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1 Overview of the Field

The investigation of eigenvalues and eigenfunctions of the Laplace operator in bounded domains, manifolds, or graphs is a subject with a history of more than two hundred years [1]. This is still a central area in mathematics, physics, engineering, and computer science and activity has increased dramatically in the past twenty years. Laplacian eigenfunctions appear as vibration modes in acoustics, as electron wave functions in quantum waveguides, or as a natural basis for constructing heat kernels in the theory of diffusion. For instance, vibration modes of a thin membrane (a drum) with a fixed boundary are given by Dirichlet Laplacian eigenfunctions, with the drum frequencies proportional to the square root of the eigenvalues. The Laplacian eigenvalue problem is archetypical in the theory of elliptic operators, while the properties of the underlying eigenfunctions have been thoroughly investigated in various mathematical and physical disciplines, including spectral theory, probability and stochastic processes, dynamical systems and quantum billiards, condensed matter physics and quantum mechanics, the theory of acoustical, optical, and quantum waveguides, and the computer sciences. Various aspects of Laplacian eigenfunctions and eigenvalues have been studied: high-frequency asymptotic behavior, isoperimetric inequalities and the related shape optimization problems, inverse spectral problems, geometric properties of nodal lines/surfaces or nodal domains. More recently, the intricate relation between the shape of the domain and the spatial distribution of eigenfunctions has been analyzed, in both high-frequency and low-frequency regimes. The intriguing properties of eigenfunctions find physical and engineering applications. For instance, the localization of some eigenfunctions in small regions of the domain was used to build noise protective walls. More generally, eigenmethods have found numerous applications in data analysis, e.g., low-dimensional parameterization of point clouds or page ranking in search algorithms (Google). Despite of a long history of investigations of Laplacian eigenfunctions, many questions remain open or have been recently formulated.

2 Recent Developments and Open Problems

A considerable progress has been recently made in understanding the geometry and topology of Laplacian eigenfunctions. For example, F. Nazarov and M. Sodin have made major advances in studying the nodal lines and domains of a random linear combination of Laplacian eigenfunctions on the sphere (i.e., spherical harmonics) belonging to the same eigenspace [2]. The interest of studying such random spherical harmonics comes from the fact that they can serve as a good model for the typical behavior of high frequency Laplacian eigenfunctions on a compact surface endowed with a smooth Riemannian metric. A. Hassell made a breakthrough in the field of quantum billiards by showing that generic stadiums are not quantum uniquely ergodic [3]. Yet another significant discovery on the structure of eigenfunctions has been made by M. Filoche and S. Mayboroda who developed a unifying mathematical approach to localization phenomena [4]. These and many other important results have been recently reviewed by one of the organizers [1]. Despite such progress, much remains open; for instance it is not understood how scaling limits of these random spherical harmonics (i.e., a random linear combination of plane waves) behave.

Beside classical spectral expansions in mathematics, Laplacian eigenfunctions turn out to be a natural tool for a broad range of areas, e.g., in data analysis to reduce dimensionality of datasets by using diffusion maps, or to study brain regions functionality. The eigenfunctions also appear as intrinsic local coordinate system with low distortion mappings.

In physics, Laplacian eigenfunctions were successfully applied for a better interpretation of nuclear magnetic resonance measurements of diffusive transport. For instance, the asymptotic properties of the heat kernel allowed experimental determination of the surface to volume ratio in porous media like sandstones or sedimentary rocks. These results are also related to Weyl asymptotic for the eigenvalues and its Weyl-Berry conjectural extension, with opening connections to number theory (e.g., spectral zeta functions).

The use of Laplacian eigenvalues as natural fingerprints to identify geometrical shapes was suggested for different applications, e.g., copyright protection, database retrieval, and quality assessment of digital data representing surfaces and solids. The related inverse spectral problems have attracted much attention in the last decades.

In this light, this workshop gathering experts in different fields, from mathematics to physics, engineering, and computer science, has provided an exciting opportunity to discuss various aspects of these long-standing problems.

3 Presentation Highlights

The presentations at this workshop can be *roughly* categorized into 7 topics although some of them are not localized in a single category: 1) Localization and geometry of eigenfunctions (Grebenkov, Maltsev, Beliaev, Filoche, David); 2) Heat/wave kernels (van den Berg, Harrell, Jakobson, Jones, Jerison); 3) Brownian motions, harmonic functions (Burdzy, Mayboroda); 4) Physical applications (Berkolaiko, Amitai, Song); 5) High dimensional data analysis, data manifolds, shape analysis (Maggioni, Bronstein, Lai, Cloninger, Saito, Beg, Meyer); 6) Shape optimization, numerical methods (Kao, Antunes, Shivakumar); and 7) Eigenvalue inequalities (Laugesen, Siudeja, Henrot, Benguria, Hermi, Mahadevan)

Disclaimer: To keep the brevity of this report, we will not include the names of the collaborators of the speakers below, which can be obtained from the abstracts of the speakers as well as their presentation slides.

3.1 Localization and geometry of eigenfunctions

Denis Grebenkov (Ecole Polytechnique) kicked off the workshop by the overview of the field of Laplacian eigenfunctions closely based on his SIAM Review paper [1]. He discussed the localization and concentration phenomena of the low-frequency Laplacian eigenfunctions due to the geometry of the domains. In particular, he focused on the exponential decay of the low-frequency eigenfunctions on domains with elongated branches, and explained a sufficient condition on the minimal length of such branches leading to localization of the low-frequency eigenfunctions.

Anna Maltsev (Bristol) discussed on localization/exponential decay of eigenfunctions of Schrödinger operator on quantum graphs. She used the so-called "Agmon metric" $\rho_E(x,y) := \int_x^y \sqrt{(V(x)-E)_+}\mathrm{d}x$ where V is the potential and E is the energy level (or eigenvalue) of the system. If $\exp(\rho_E(0,x))\psi \in L^2$ is shown, then ψ clearly decays exponentially. She started from the review of the known results for path graphs and discussed her approach to generalize them for rooted trees, regular trees, and finally general trees. She concluded her talk with an example of "harmonic millipede" graphs and some open problems, e.g., how to tackle more general graphs containing cycles.

Dmitry Beliaev (Oxford) talked about the nodal geometry of random plane waves, which behaves in a similar manner as that of high-energy eigenfunctions in stadium-shape domains that often appear in the quantum chaos literature. In particular, he reviewed the above Nazarov-Sodin result on the estimate of the number of nodal domains generated by random linear combinations of spherical harmonics of degree n (forming $2n + 1$-dimensional eigenspaces). Moreover, he presented numerical results that suggest a way to amend the Bogomolny-Schmit conjecture on the percolation model by considering the random graphs generated by random plane waves on the square lattice.

Both Filoche and David talked about their program that led to another small-group workshop, *Focused Research Group: Localization of Eigenfunctions of Elliptic Operators* that was held immediately after our workshop. Marcel Filoche (Ecole Polytechnique) discussed the localization of eigenfunctions of both the Laplace and biharmonic operators on complicated domains including some slits and punctures. His discussion on the "inverse" problems, i.e., where to put punctures on a thin plate in order to form a desired localized pattern, was interesting and important for real-life applications. In addition, he described his work on the Anderson localization (i.e., the Schrödinger eigenvalue problems with random potentials), and the conditions under which the delocalization occurs. The key tool he used for all these is the so-called "landscape" function, i.e., the solution of $\mathcal{L}u = 1$ with the appropriate boundary condition where $\mathcal{L}$ is an elliptic partial differential operator (e.g., Laplace, biharmonic, Schrödinger, etc.). His numerical simulations agreed well with the physical experiments conducted at the Langevin Institute.

Guy David (Paris-Sud) discussed a related free boundary problem: how to decompose a given domain into N subdomains via minimization of a certain functional that includes a usual energy term and a term penalizing subdomains of unbalanced sizes. It turns out that the Laplacian/Schrödinger eigenfunctions tend to localize on such subdomains, which were investigated further in the above-mentioned FRG.

3.2 Heat and wave kernels

Michiel van den Berg (Bristol) discussed his work on heat flow on Riemannian manifolds. He started off with his earlier results on heat content asymptotics on a simpler case of a compact Riemannian manifold with C^∞ boundary, then discussed the case of a compact Riemannian manifold without boundary before reaching his latest results on a complete, non-compact Riemannian manifold with non-negative Ricci curvature. In particular, he discussed heat flow from a subdomain Ω of such a manifold M into $M \setminus \Omega$ if the initial temperature distribution is the characteristic function of Ω. For $|\Omega| = \infty$, he obtained a necessary and sufficient condition to have finite heat content for all $t > 0$ and also obtained upper- and lower bounds for the heat content. Two-sided bounds are obtained for the heat loss of Ω in M if $|\Omega| < \infty$.

Evans Harrell (Georgia Tech) presented two rather distinct results whose common theme is a certain optimality about heat traces. The first one is an optimal placement problem for an obstacle in a domain so as to maximize or minimize the heat trace. More specifically, given a region Ω, excluding a subset B of a fixed shape (in practice round) at an unspecified position, how the extremal values of an eigenvalue or other spectral function can be achieved by moving B around? In the case of the Dirichlet boundary condition, the maximum is achieved when B is in a distinguished subset while the minimum is attained when B touches the boundary $\partial\Omega$ for some class of regions Ω (e.g., annular regions). These results are mainly due to Harrell-Kröger-Kurata (2001), but he also traced some of more recent works done by the others. The second topic of his talk was a set of sharp semiclassical inequalities for sums of eigenvalues, which are closely related to traces of heat kernels via Karamata's theorem and its variants. These are based on a new variational principle that incorporates averages of the lowest k eigenvalues, which implies sharp estimates for Laplacians of various kinds, including those on graphs and on quantum graphs.

Dmitry Jakobson (McGill) presented his construction of Gaussian measures on the manifold of Riemannian metrics with the fixed volume form. He showed that diameter, Laplacian eigenvalue and volume entropy functionals are all integrable with respect to these measures. He also presented his computation of the characteristic function for the L^2 (also called Ebin) distance from a random metric to the reference metric.

Peter Jones (Yale) discussed the importance of randomness that often gives rise to certain orders and structures. In particular, he discussed the notion of Gaussian free fields (mainly in 2D) and contrasted their differences from the Brownian motion, and explained that the cosmic microwave background may be viewed as a realization of such Gaussian free field.

David Jerison (MIT) discussed his work on wave trace: a spectral invariant function $W(t) := \sum_j \cos(\lambda_j t)$, in which the sum is over all eigenvalues λ_j^2 of $-\Delta$ on a compact Riemannian manifold. It is called the wave trace because it is the trace of the solution operator for the wave equation. It is well known that $W(t)$ is singular at times t that are equal to the length of geodesics. The singularities are well understood if the length is an isolated point in the time line. He presented his theorem in the case of the unit disk in $\mathbb{R}^2$ with the Dirichlet boundary condition: $W \in C^\infty[2\pi, 8)$.

3.3 Brownian motions, harmonic functions

Chris Burdzy (Univ. Washington) discussed his recent results on obliquely-reflected Brownian motion in the unit disk in $\mathbb{R}^2$ and some fractal domains. Dealing with obliquely-reflected Brownian motion is tough since classical Dirichlet form approach to Markov processes is limited to symmetric processes and obliquely-reflected Brownian motion is not symmetric. Also, such a Brownian motion could jump along the boundary in the tangential direction if it hits the boundary with certain angles. He showed his theorem stating that for such a Brownian motion X_t, $(1/t)\arg X_t - \mu$ converges to the Cauchy distribution as $t \to \infty$ where μ is a rate of rotation of this Brownian motion. For fractal domains, he presented two possible approaches: the one based on smooth approximation of a domain and the other based on conformal mapping.

Svitlana Mayboroda (Univ. Minnesota) discussed the relationship between uniform rectifiability and harmonic functions. In relatively friendly geometric settings, e.g., on Lipschitz domains, harmonic measure is absolutely continuous w.r.t. the Lebesgue measure. Moreover, quantitative absolute continuity of elliptic measure is equivalent to solvability of the Dirichlet problem in L^p, square function estimates, Carleson measure estimates, and a certain approximation property, so called ϵ-approximability of solutions. In particular, she addressed the following question: what are the key geometrical properties of the boundary responsible for such a behavior of harmonic functions and whether these can be generalized for uniformly rectifiable domains without assuming the connectivity, whose answer is 'Yes.'

3.4 Physical applications

Gregory Berkolaiko (Texas A & M Univ.) talked about the so-called Dirac points in the spectra of graphene (honeycomb) structure. Many exciting physical properties of graphene can be traced to the presence of conical singularities ("Dirac points") in its dispersion relation. He first traced a history of the proof of the presence of such Dirac points in the context of Laplacian on domains in $\mathbb{R}^2$, discrete Laplacian, and quantum graph Laplacian. He then presented his general yet very simple proof that works in all the above models. His proof of the presence of the Dirac points uses quotients of the operator by the (co)representations of the symmetry group, and a proof of stability of Dirac points uses the so-called Berry phase: a phase gained by an eigenfunction after the parameters specifying that eigenfunction rotate around a contour in the parameter space. He also showed some animations to illustrate these ideas.

Assaf Amitai (MIT) presented his results on the mean first encounter time (MFET) between two polymer sites, which is an interesting application of a Brownian motion in high dimensional manifolds. The encounter between two sites on chromosomes in the cell nucleus can trigger gene regulation, exchange of genetic material, and repair of DNA breaks. By forming a DNA loop, a protein located on the DNA can trigger the expression of a gene located far along the chain. He computed asymptotically the MFET between the head and tail of the polymer as a function of the distance ϵ between them, using the classical Rouse polymer model, in which the polymer is described as a collection of bead monomers connected by harmonic springs. This novel asymptotic relies on the expansion of the spectrum of the Fokker-Planck operator as a function of ϵ, and the explicit computation of the Riemannian volume for Chavel-Feldman formula, which gives the shift in the spectrum of the Laplace operator when Dirichlet boundary conditions are imposed on the boundary of tubular neighborhood of a constraint manifold (removal of a manifold with a small volume).

Yiqiao Song (Schlumberger-Doll Research & Mass. Gen. Hospital, Cambridge, MA), who was the only participant from industry, talked about diffusion dynamics from the viewpoint of multi-point correlation functions, and its applications to study the microstructure of a wide range of porous materials, including geological formations and biological tissues. In order to characterize the non-Gaussian diffusive behavior in heterogeneous media, the fourth order cumulant (kurtosis) can be expressed through the 4-point correlation function which allows one to distinguish contributions from pore size distribution. The related NMR experiments on asparagus and avocados were discussed. In the second part of the talk, he presented the concepts of Diffusion Eigenmode Spectroscopy. This experimental tool relies on the properties of Laplacian eigenfunctions with the Robin boundary condition. By manipulating the nuclei magnetization with magnetic fields, one can probe the geometric properties of a porous medium through the measurable relaxation curves.

3.5 High-dimensional data analysis, manifold learning, shape analysis

Mauro Maggioni (Duke) discussed a geometry-based statistical learning framework for performing model reduction and modeling of stochastic high-dimensional dynamical systems. He considered two complementary settings. In the first one, given long trajectories of a dynamical system (e.g., molecular dynamics governed by a Fokker-Planck equation), he presented new techniques for estimating in a robust fashion: (i) an effective number of degrees of freedom of the system; and (ii) a local scale where the dynamics is well-approximated by a reduced dynamics with a small number of degrees of freedom. He then used these ideas to produce an approximation to the generator of the system and obtain, via eigenfunctions, reaction coordinates for the system that captures the large time behavior of the dynamics. In his second setting, the only available data are short-time/local simulations (of various different stages) of a high-dimensional stochastic system due to the often expensive nature of the numerical simulators. Then, he introduced a statistical learning framework for determining automatically a family of local approximations to the system. He then presented applications of this method including homogenization of rough diffusions and deterministic chaotic systems in high-dimensions.

Michael Bronstein (Univ. Lugano, Switzerland) discussed his long-time and large-scale project on manifold correspondence, a fundamental and notoriously hard problem with a wide range of applications in geometric processing, graphics, computer vision, and machine learning. He started off with the classical method, i.e., decomposing each given object into a linear combination of the Laplace-Beltrami eigenfunctions, and considering the correspondence between two different eigen-coordinate systems. Unfortunately, those eigenbasis methods suffer from several issues such as sign ambiguities of eigenfunctions and the differences in eigenfunctions for non-isometric manifolds, which are more pronounced for the high-frequency eigenfunctions. In order to solve these problems, he discussed the use of coupled (or joint) diagonalization of two sets of eigenbases for drastic improvement for the correspondence problems of two non-isometric objects (e.g., an elephant vs a horse). Finally, he discussed a method to estimate the matrix describing the deformation from one manifold to the other based on geometric matrix completion under the sparsity and smoothness constraints. This method does not require any computation of eigenbases. He demonstrated all of these methods using many 3D geometric objects.

Rongjie Lai (RPI) discussed his work on processing and analysis of point clouds using Laplace-Beltrami operator. Point clouds are the simplest and most basic forms for data representation in 3D modeling, imaging science, the Internet and many others. Although raw data appear as an unstructured and unorganized set of points, they are usually with certain coherent structures which allow one to model them as points sampled from lower dimensional Riemannian manifolds in a high dimensional space. Analyzing and inferring the underlying structure from the point clouds are crucial in many applications. Lai discussed his work on solving PDEs on point clouds, which requires only local information such as k nearest neighbors of each point. In particular, the Laplace-Beltrami eigenfunctions can be used to embed the high-dimensional point clouds onto a low-dimensional Euclidean space where one can often see the global organization of these point clouds. Again, similar to what Bronstein listed, the difficulties using such Laplace-Beltrami eigenfunctions are: their sign ambiguities, ambiguities due to the multiplicities of the corresponding eigenvalues (if any), and natural ordering of the eigenfunctions. In order to resolve those difficulties, he incorporated the idea of optimal transportation after computing the two sets of the Laplace-Beltrami eigensystems. He also discussed a few possible ideas to speed up the computation of such an optimization problem utilizing the intrinsic multiscale nature of the eigenfunctions.

Alexander Cloninger (Yale) discussed ways to augment diffusion kernels on datasets using knowledge of function values on a subset of the data. An external function on the dataset, whose random and finite realizations (e.g., noisy versions of the true function values) are available to the user, is used to discover features in the data that are locally invariant as well as features that are locally insignificant, and to build a diffusion kernel on the data that diffuses quickly along these local irrelevant features. The new metric and diffusion kernel, generated via iterations of stacked neural nets, create an embedding which captures the geometry of the data while yielding a low Lipschitz constant with respect to the functions of interest. Along with this, the new kernel does not depend explicitly on the function values, which makes it trivially extendable to new points. He demonstrated the performance of their algorithms using synthetic datasets as well as the real medical dataset. The latter is quite intriguing: the dataset consists of about 80 different measures of more than 1,000 hospitals in US, e.g., the frequency of prescribing aspirin hospitalization for heart attack; time spent in the emergency department prior to discharge home; the patients' experience and comments on their doctors; patients' mortality at 30 days after hospitalization for heart failure, etc. From this dataset, he successfully characterized the overall hospital performance as well as the ranking and grouping of these hospitals.

Naoki Saito (UC Davis) discussed his work on Laplacian eigenfunctions that 'do not feel the boundary.' These are in fact the eigenfunctions of the integral operators commuting with the Laplacian on domains in $\mathbb{R}^d$ and their discretized versions. These eigenfunctions satisfy the Helmholtz equation inside the domain, and can be extended smoothly and harmonically outside of the domain. The kernel of these integral operators are the so-called free space Green's functions or the fundamental solution of the Laplacians. He discussed the similarities and differences of these eigenfunctions with the so-called Krein-von Neumann Laplacian eigenfunctions. Also, he described the intimate relationship between the discretized version of these integral operators on the lattice graphs, their distance matrices, and their graph Laplacians, and in particular, showed that the eigenvectors of the graph Laplacian of a lattice graph coincide with those of the distance matrix when it is sandwiched by the projector onto the orthogonal complement of the DC component. He concluded his talk by raising a natural question: what is the corresponding integral operator (or the kernel matrix) that commutes with the graph Laplacian of a general graph.

Faisal Beg (Simon Fraser Univ.) and his postdoc Karteek Popuri talked about their work on medical image analysis and computational anatomy using Laplacian eigenfunctions and heat kernels. They reviewed the use of the ratios of the Laplacian eigenvalues as a feature vector for classifying hippocampus shapes (obtained from MRI images) into those of Alzheimer's Disease patients and those of control patients. Moreover, the nodal lines of the eigenfunctions are also used for partitioning various anatomical manifolds. As for the use of heat kernels, they discussed cortical thickness smoothing, which is similar in spirit to the popular nonlocal means denoising algorithm. They concluded their talk by posing two open problems: 1) how to capture cortical structure beyond thickness; 2) how to capture and characterize vascular structures.

François Meyer (Univ. Colorado, Boulder) talked about his work on prediction of evolution of epilepsy from electrophysiological measurements around hippocampuses of 23 rats. His hypothesis is that the response to an auditory stimulus changes during epileptogenesis. Hence, he needed to quantify changes in the auditory response that are correlated to changes in the hippocampus associated with epileptogenesis. The challenge is to decode the progression of the disease from the auditory response. To do so, he first computed the stationary wavelet transform of the successive segments of electrophysiological responses of rats to auditory stimuli. Then, he chose a subset of those wavelet coefficients that best separate the four conditions of rats under the forced epileptogenesis experiments: baseline, silent, latent, and chronic. Then, he tracked the evolution of these coefficients in the phase space by embedding these high-dimensional coefficients in $\mathbb{R}^d$ where d is much smaller than the number of those coefficients followed by training the hidden Markov model, which in the end estimates the probability of being in one of the four states listed above. His result was quite impressive: his algorithm reliably predicted eventual spontaneous recurring seizures. In rats that did not develop spontaneous recurring seizures, his algorithm predicted a return to baseline.

3.6 Shape optimization, numerical methods

Chiu-Yen Kao (Claremont McKenna College) talked about her work on shape optimization for eigenvalue problem involving biharmonic operators. In particular, she focused on the vibration of a plate with the nonhomogeneous thickness or density under the clamped ($u = 0 = \partial_\nu u$) or the simply-supported ($u = 0 = (\partial_\nu^2 + \partial_\tau^2)u$) boundary conditions. Her optimization problem is as follows: for a given domain shape under one of the above two boundary conditions, find the thickness or density distribution of the plate material that minimizes or maximizes the lowest eigenvalue. Her numerical scheme iterates between the two stages: 1) computation of the optimal eigenpair under the fixed density and 2) computation of the optimal density under the fixed eigenfunction. This process converges to the optimal density ρ^* that gives the extremal λ_1. The same iterative scheme also works for material thickness. She demonstrated the efficiency and robustness of her numerical schemes using a variety of examples including square, disk, annulus in 2D. It is interesting to note that the optimal thickness distribution on the 2D annulus under the simply-supported boundary condition turns out to be symmetric eigenfunction u_1 if the inner radius is small, but become asymmetric if the inner radius becomes large (i.e., the annulus becomes thin).

Pedro Antunes (Univ. Lisbon, Portugal) presented his numerical scheme and his results for the shape optimization problem associated with localized Laplacian eigenfunctions. Motivated by the Shnirelman theorem on the quantum ergodicity, his aim was to provide a numerical scheme for the following shape optimization problem: given a planar domain V with $|V| < 1$, find a *convex* shape Ω that minimizes (or maximizes) the energy concentration $\|u_j\|^2_{L^2(V)}$ on V subject to $V \subset \Omega$ and $|\Omega| = 1$ where u_j is the jth Dirichlet-Laplacian eigenfunction on Ω, $j \in \mathbb{N}$. His method uses a clever parameterization of the boundary curve $\partial\Omega$, the so-called Hadamard shape derivatives, and the Method of Fundamental Solutions (MFS). His numerical results nicely confirmed the theory of Harrell, i.e., the minimization of the energy concentration $\|u_1\|^2_{L^2(V)}$ is achieved when V touches Ω while the maximum is achieved when V is centered in Ω and away from $\partial\Omega$. He also obtained those Ω's for different eigenfunctions u_j with $j = 23, 160, 200$, etc. for various fixed V.

Pappur N. Shivakumar (Univ. Manitoba) discussed infinite linear algebraic systems with diagonally dominant matrices and its important application, e.g., computing the Dirichlet-Laplacian eigenvalues of various planar domains. His approach first represents a domain with biaxial symmetry by an analytic curve in the complex plain as a power series of $z\overline{z}$ or $z + \overline{z}$. Such a representation converts the Dirichlet-Laplacian eigenvalue problem to an infinite system of linear equations whose coefficients are polynomials of the eigenvalues depending on the domain shape. The zeros of the determinant of this linear system determine the eigenvalues. To compute the eigenvalues numerically, he truncated the infinite system matrix and demonstrated that this strategy generated extremely good approximation to the eigenvalues for various domains. He finally illuminated the classical question "Can you hear the shape of a drum," by his constructive analytic approach: a pre-knowledge of eigenvalues yields the information about the boundary for many domains with analytic boundary curves (e.g., circle, ellipsis, annulus, etc.) while in the case of a square, this approach fails. This may indicate why the non-analytic (i.e., polygonal) boundary cases could be the counter-examples of Kac's question.

3.7 Eigenvalue inequalities

Rick Laugesen (Univ. Illinois, Urbana-Champaign) gave a talk on Steklov spectral inequalities through quasiconformal mapping. The Steklov eigenvalue problem is an eigenvalue problem with the spectral parameter in the boundary conditions while the governing equation itself is the Laplace equation. Vibration of a free membrane whose mass is concentrated at the boundary is a typical example. Recently, there has been a growing interest in the Steklov problem from the viewpoint of spectral geometry as well as its applications including liquid sloshing. Its spectrum coincides with that of the Dirichlet-to-Neumann operator. He reviewed the basics of the Steklov eigenvalue problems including a question "can you hear the shape of a Steklov membrane?" The answer might be 'yes' and so far, no counterexamples are known. Things that can be heard include boundary length (by Weyl) and the number of boundary components (by Girouard et al. 2014). He then showed that the disk maximizes various functionals of the Steklov eigenvalues, under normalization of the perimeter and a kind of boundary moment. The results cover the first eigenvalue, spectral zeta function and trace of the heat kernel.

Bartłomiej Siudeja (Univ. Oregon) discussed his work on nearly radial Neumann modes on highly symmetric domains. Schiffer's conjecture states that if a Neumann eigenfunction is constant on the boundary of a domain, then either the eigenfunction is constant on the whole domain, or the domain is a disk. The disk is special, due to the presence of radial modes. He discussed the existence of Neumann modes on regular polygons and boxes which are nearly radial (do not change sign on the boundary). Siudeja got a very interesting result: if the domain is a regular polygon with more than 4 sides, then there exists an eigenfunction that is strictly positive on the boundary but not constant inside the domain. He then also discussed his recent work on the 3D Steklov eigenvalue problem with the Neumann boundary condition at the sides of the container. In particular, the so-called "high-spots" conjecture (the highest spot of the second eigenfunction is on the boundary of the container) was discussed. He showed two results: (i) it is possible to slosh a fluid so that the liquid surface moves in unison on the boundary of a cylindrical cup whose base is a regular polygon with more than 4 sides; (ii) in the case of the triangular base, the fluid moves up in one place on the boundary and moves down in another on the boundary.

Antoine Henrot (Univ. Lorraine) discussed how to maximize the first eigenvalue of the Laplacian with Dirichlet boundary conditions by placing some obstacle K inside a fixed domain Ω. He first reviewed some known results for that problem, mainly when the shape of the obstacle K is already given (then the question is to find its location), which was also discussed by Harrell. Here the shape of K is free but connected and with a fixed perimeter. He proved the existence of a maximizer. He also discussed more specific case where Ω is a ball, for which he proved that the obstacle should be a concentric ball.

Rafael Benguria (Pontificia Universidad Católica de Chile) discussed the Brezis–Nirenberg problem on $\mathbb{S}^n$ in spaces of fractional dimension, in particular, $2 < n < 4$. This is a nonlinear eigenvalue problem described as:

$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. This problem has its origin in the Lane-Emden equation, a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. In 2002, Bandle and Benguria identified the range of the eigenvalues, which depends on the geodesic radius of the ball, where this problem has a unique positive solution in the case of $n = 3$. Then, he discussed his recent result for the case $2 < n < 4$ where he also identified such eigenvalue range that depends on the indices of the associated Legendre functions.

Lotfi Hermi (Univ. Arizona) discussed the isoperimetric upper bound for the fundamental tone of the membrane problem for a class of wedge-like domains. Such upper bound can be used to estimate the fundamental tone of a right-angle triangular membrane or more generally regular α-polygon with n sides. Moreover, he proposed a new lower bound for its "relative torsional rigidity" of such domains, i.e., the torque required for unit angle of twist per unit length when the shear modulus is 1 relative to the weight function (e.g., density function of samples over manifolds, etc.).

Rajesh Mahadevan (Universidad de Concepcion, Chile) discussed eigenvalue minimization for the clamped plate under compression, $(\varDelta^2 + \tau\varDelta)u = \lambda u$ in Ω with $u = |\nabla u| = 0$ on $\partial\Omega$. In particular, he addressed the isoperimetric problem: among domains of equal volume, for which domain the first eigenvalue of the clamped plate with compression is a minimum? In the case of no compression (i.e., $\tau = 0$), Rayleigh, in 1877, conjectured that the ball is the minimizer of this problem. The Rayleigh conjecture for the clamped plate with $\tau = 0$ has been proved by Nadirashvili for $d = 2$ (1995) and Ashbaugh and Benguria for $d = 3$ (1995). He demonstrated that for the small positive values of τ, the first eigenvalue is a minimum when the domain is a ball.

4 Scientific Progress Made

In joint work carried out during the workshop, Laugesen and Siudeja answered a quested of J. Cima about the location of the maximum points of ground state solutions of semilinear Poisson equations. Specifically, they found examples in which the maximum points of two different equations are in two different locations albeit remarkably close together (differing only on the scale of 10^{-4}). These researchers also developed a road map for proving a Weyl Law for the Steklov spectrum on piecewise smooth domains (work in progress, initiated during the workshop).

Beliaev, Grebenkov, and Jones have completed their joint work on multiscale representations of Gaussian processes and fields. Antunes and Grebenkov have started a new project on localization of Laplacian eigenfunctions in three-dimensional domains, which theoretical exponential estimates will be combined with the efficient numerical method of fundamental solutions.

David, Filoche, Jerison, Mayboroda, together with Doug Arnold, organized an extended focused research group workshop on localization of Laplacian eigenfunctions after our workshop.

5 Outcome of the Meeting

We feel this workshop was a success. We also believe that more collaborations have been established, particularly, between scientists from pure and applied fields. For instance, the workshop participants working on numerics of the eigenfunctions such as Pedro Antunes seem to be on high demand for revealing and checking mathematical hypotheses. Through the presentations and interactions during this workshop, the participants also realized that they encountered similar problems in their own research and shared their experiences and discussed potential solutions. Examples include ordering and sign ambiguities of eigenfunctions by people working in data manifolds; shape optimization associated with obstacle residing within a given domain. eigenfunction localizations and how to control them; and the other extensions to the traditional Laplacian eigenvalue problems, e.g., biharmonic eigenvalue problems, Steklov eigenvalue problems, nonlinear eigenvalue problems.

We also discussed a possibility to propose a semester-based long program at ICERM (Brown Univ.) or IPAM (UCLA), as a continuation of our program. Peter Jones, knowing both places firsthand, explained the operational mechanisms of these institutes, and discussed how to write proposals for short and long programs.

We also got many positive comments from the participants, some of which are (for all the comments, see the BIRS testimonial website):

> "Always a pleasure to be at Banff. I heard for the first time the talks by Mayboroda, Filoche, David and Jerison on eigenfunction concentration, that was quite inspiring, I will probably give a survey lecture on this a few times in the next few months. I also started a new paper with Pedro Antunes from Lisboa at the conference, not sure how long it will take." — D. Jakobsen

> "The Meeting was a wonderfully stimulating event, bringing together researchers who use a wide variety of techniques to approach problems in the same field." — R. Laugesen

> "During the workshop I have had the occasion to work with Michiel van den Berg on some questions related to the first eigenvalue of the Dirichlet-Laplacian and the torsion (in particular we are interested in getting sharp lower bounds for the product of these two natural quantities). I also discussed with Bartek Siudeja and Rick Laugesen in the one hand and Pedro Antunes on the other hand about a project of a collective book on recent results of spectral theory." — A. Henrot

> "I found the conference very stimulating. I had a chance to talk to some people who I know well, for example, Michiel van den Berg, Peter Jones, Marcel Filoche and Bartek Siudeja. I have also renewed some friendships, for example, with Antoine Henrot, and make new friends, for example, with Dima Jakobson. It was great to see what is done in fields that are somewhat different from mine. I learned about the relationship between eigenfunctions and billiards, and about results and conjectures on the Gaussian character of the value function for high frequency eigenfunctions. Overall, the idea of bringing people in different fields to one conference was excellent." — C. Burdzy

> "I found last week's workshop at BIRS very stimulating: the range of topics was quite broad, and the talks were on the whole of good quality. It also allowed me to work with some colleagues notably Antoine Henrot. However, I also had some interesting discussions with several other participants. The facilities at BIRS are first class, and the surrounding scenery spectacular. This was my second time at BIRS, the first visit was in 2004. I very much hope it won't be my last!" — M. van den Berg

> "This was one of the most interesting conferences I have attended in years, and I believe it might bring some profound insights to my research. I was also able to make contacts with great people and seed new collaborations." — M. Bronstein

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