Rigidity for grid-like reflection frameworks

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Advances in Combinatorial and Geometric Rigidity
BIRS, 13-17th July 2015

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- What form does the infinitesimal flex condition take?
- Is infinitesimal rigidity a generic property?

The rigidity map for $G = (V, E)$ and $(X, \|\cdot\|)$ is defined by,

$$f_G : X^{|V|} \to \mathbb{R}^{|E|}, \quad (x_v)_{v \in V} \mapsto (\|x_v - x_w\|)_{vw \in E}.$$

An infinitesimal flex for (G, p) is a vector $u \in X^{|V|}$ such that

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$\mathcal{F}(G,p)$:= vector space of all infinitesimal flexes of (G,p).

$\mathcal{T}(G,p)$:= vector subspace of all trivial infinitesimal flexes.

A framework (G,p) is infinitesimally rigid if $\mathcal{F}(G,p) = \mathcal{T}(G,p)$.

A norm on a real vector space X is a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies the conditions

(i) $\|x\| \geq 0$ for all $x \in X$, and, $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and all $\lambda \in \mathbb{R}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Eg. some norms on $\mathbb{R}^d$,

- $\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}}$, $1 < p < \infty$.
- $\|x\|_1 = \sum_{i=1}^d |x_i|$ and $\|x\|_\infty = \max_{i=1,2,\ldots,d} |x_i|$
- $\|x\|_{\mathcal{P}} = \begin{cases} 2|x_1| & \text{if } |x_1| \leq |x_2| \\ |x_1| + |x_2| & \text{if } |x_1| \geq |x_2| \end{cases}$

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$$(p_v - p_w)^{(q-1)} \cdot (u_v - u_w) = 0, \quad \forall\, vw \in E$$

where $x^{(q-1)} := (\mathrm{sgn}(x_1)|x_1|^{q-1}, \ldots, \mathrm{sgn}(x_d)|x_d|^{q-1})$.

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- polyhedral norms,

$$\hat{F}_{vw} \cdot (u_v - u_w) = 0, \quad \forall\, vw \in E$$

  where F_{vw} is an associated facet of the unit ball.

The flex condition in general:

$$\varphi_{v,w}(u_v - u_w) = 0, \quad \forall\, vw \in E$$

where $\varphi_{v,w} : X \to \mathbb{R}$ is the linear functional,

$$\varphi_{v,w}(x) := \lim_{t\to 0} \frac{1}{t}\left(\|p_v - p_w + tx\| - \|p_v - p_w\|\right).$$

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- $\sup_{\|x\|\leq 1} |\varphi_{v,w}(x)| = 1$.
- $\varphi_{v,w}(p_v - p_w) = \|p_v - p_w\|$.
- $\varphi_{v,w}$ is a support functional for $\frac{p_v - p_w}{\|p_v - p_w\|}$.

Proposition

If the rigidity map f_G is differentiable at p then

(i) $df_G(p)u = (\,\varphi_{v,w}(u_v - u_w)\,)_{vw\in E}$ *for all* $u \in X^{|V|}$.

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(Assume f_G is differentiable at p from here on...).

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- How does the rigidity operator decompose?
- What can be deduced from the gain graph G/Γ?

Decomposition into symmetric and anti-symmetric parts:

Proposition

If (G,p) *is* $\mathbb{Z}_2$*-symmetric in* $(X, \|\cdot\|)$ *then,*

(i) $df_G(p) = R_1 \oplus R_2$.

(ii) $\mathcal{F}(G,p) = \mathcal{F}_1(G,p) \oplus \mathcal{F}_2(G,p)$.

(iii) $\mathcal{T}(G,p) = \mathcal{T}_1(G,p) \oplus \mathcal{T}_2(G,p)$.

If $\|\cdot\|_{\mathcal{P}}$ is a polyhedral norm and F is a facet of the unit ball $\mathcal{P}$ then there exists a unique extreme point $\hat{F}$ in the polar set $\mathcal{P}^{\triangle}$ such that

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$$\varphi_{v,w}(x) = \hat{F} \cdot x, \quad \forall\, x \in X,$$

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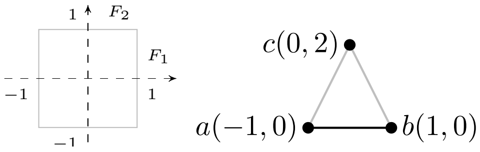
Denote by G_F the monochrome subgraph of G spanned by edges $vw \in E$ with framework colour $[F]$.

Theorem (K - Power, 2014)

A grid-like framework (G, p) is isostatic if and only if the induced monochrome subgraphs G_{F_1} and G_{F_2} are both spanning trees in G.



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A subgraph of G_0 for which every connected component contains exactly one cycle, each of which is unbalanced, is called an unbalanced map graph in G_0.

Theorem (Symmetrically isostatic frameworks)

Let (G, p) be a grid-like reflection framework with $G \neq K_2$. If the reflection acts freely on V then TFAE:

(i) *(G, p) is symmetrically isostatic.*

(ii) *$G_{F_1,0}$ is an unbalanced spanning map graph in G_0 and $G_{F_2,0}$ is a spanning tree in G_0.*

(a) (b) (c) (d)

Figure: A symmetrically isostatic (but not anti-symmetrically isostatic) reflection framework in $(\mathbb{R}^2, \|\cdot\|_\infty)$ (a) and its signed quotient graph (G_0, ψ) (b). An anti-symmetrically isostatic (but not symmetrically isostatic) reflection framework in $(\mathbb{R}^2, \|\cdot\|_\infty)$ (c) with the same signed quotient graph (G_0, ψ).

Corollary (Rigid frameworks with reflectional symmetry)

Let (G, p) be a grid-like reflection framework with $G \neq K_2$. If the reflection acts freely on V then TFAE:

- (i) (G, p) *is rigid.*
- (ii) G_0 *contains a spanning subgraph* H_0 *such that the monochrome subgraphs* $H_{F_1,0}$ *and* $H_{F_2,0}$ *are both connected unbalanced spanning map graphs.*

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Problem: Given a graph G and a group action $\theta : \Gamma \to \mathrm{Aut}(G)$, determine whether there exists $p \in X^{|V|}$ and a representation $\tau : \Gamma \to \mathrm{Isom}(X, \|\cdot\|)$ such that (G, p) is rigid in $(X, \|\cdot\|)$ and Γ-symmetric (w.r.t. θ and τ).

Theorem (DK - B Schulze, 2014)

Let $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ be a group action on G where $\mathbb{Z}_2 = \langle s \rangle$. TFAE:

(i) *There exists p such that (G, p) is an isostatic grid-like framework with reflectional symmetry.*

(ii) *G is a union of two edge-disjoint spanning trees, both of which are $\mathbb{Z}_2$-symmetric with respect to θ, and $|E_s| = 0$.*

Theorem (Symmetrically isostatic graphs)

Let G be a $\mathbb{Z}_2$-symmetric graph. If the action is free on V then TFAE:

(i) *There exists p such that (G, p) is a symmetrically isostatic grid-like framework with reflectional symmetry.*

(ii) *The gain graph (G_0, ψ) is $(2, 2, 1)$-gain tight.*

- D.K., S.C. Power, *Infinitesimal rigidity for non-Euclidean bar-joint frameworks*, Bull. Lond. Math. Soc. 46 (2014), no. 4, 685-697.
- D.K., *Finite and infinitesimal rigidity with polyhedral norms*, Discrete Comput. Geom. (2015)
- D.K., B. Schulze, *Maxwell-Laman counts for bar-joint frameworks in normed spaces*, Linear Algebra Appl. 481 (2015), 313-329.
- D.K., S.C. Power, *The rigidity of infinite graphs*, arXiv:1310.1860 (2013)
- D.K., B. Schulze, *Rigidity characterisations for graphs with two edge-disjoint symmetric spanning trees*, arXiv:1408.4637 (2014)
- D.K., B. Schulze, *Motions of grid-like reflection frameworks*, preprint (2015)

Thank you