On Flattenability of Graphs

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Flattenability

A graph G is *d-flattenable* if for every linkage (bar-lengths in a given norm) that has a realization in some dimension, that linkage also has a realization in R^d.[1]

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How to show non-d-flattenability?

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A Useful Theorem

This was shown in Sitharam-Gao[2] for the l_2 norm:

Theorem

For any l_p norm, a graph G is d-flattenable iff G the set of attainable l_p^p edge-length vectors for G in d-dimensions is convex [3].

[2] M. Sitharam and H. Gao, "Characterizing graphs with convex and connected cayley configuration spaces," *Discrete & Computational Geometry*, vol. 43, no. 3, pp. 594–625, 2010.

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If the attainable l_p^p edge-length vectors are not convex, then G is not d-flattenable.

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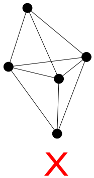
l_1 2-flattenability

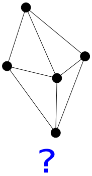
K_4 is 2-flattenable under l_1[4]

For 5 vertex graphs, we have shown the following:

- K_5 is not 2-flattenable (Known)
- Banana is not 2-flattenable (Uses previous result)
- 4-Wheel is unknown (OPEN)
- All others are 2-flattenable

If 4-wheel is not 2-flattenable, it is the only forbidden minor



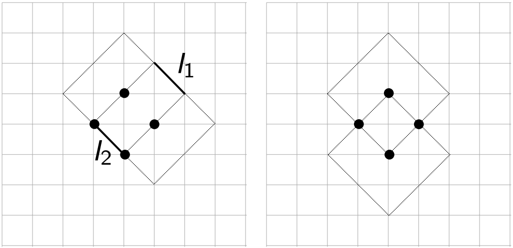


[4] H. Witsenhausen, "Minimum dimension embedding of finite metric spaces," *Journal of Combinatorial Theory, Series A*, vol. 42, no. 2, pp. 184–199, 1986.

Banana is not 2-flattenable

Consider a banana with unit length edges for all except one edge.

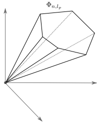
K_4 with unit lengths is a subgraph. The only realization is 4 points at corners of the unit ball.



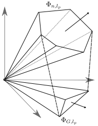
In the above, the final vertex can go at the intersection of the unit balls shown. Taking both them, we get a non-convex set for the lengths of the non-edge

Some Definitions

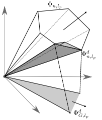
The *cone* [5] of all pairwise l_p^p-distance vectors on n-point configurations: Φ_{n,l_p}



The *d-dimensional stratum*: pairwise distance vectors of d-dimensional point configurations: Φ^d_{n,l_p}



The projection or shadow of this cone on an edge set G: Φ_{G,l_p}



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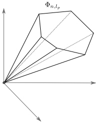

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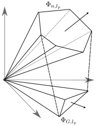

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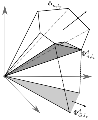

Theorem (Restatement)

G is d-flattenable iff Φ^d_{G,l_p} is convex

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Proof Idea

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The "only if" direction is immediate. Note that $\Phi_{G,l_p} = \Phi^d_{G,l_p}$ iff G is d-flattenable.

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For the "if" direction, we need a smaller result:

- Φ_{n,l_p} is the convex hull of the d dimensional stratum for any d.
- The projection of the convex hull of Φ^d_{n,l_p} onto G is the convex hull of Φ^d_{G,l_p}

Big Question

We want to know if flattenability is a generic property.

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Definition

An l_p framework (G, r) is *generic* w.r.t d-flattenability if (i) there is an open neighborhood Ω of the corresponding distance vector δ_r in the *relative interior* of the cone Φ_{n,l_p} and (ii) (G, r) is d-flattenable iff all frameworks of Ω are.

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Note: a property P is *generic* if *all* generic frameworks simultaneously satisfy or do not satisfy P.

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Note: a property P is *generic* if *all* generic frameworks simultaneously satisfy or do not satisfy P.

In other words: If $\exists$ a generic d-flattenable linkage, is G d-flattenable (Connelly Question)?

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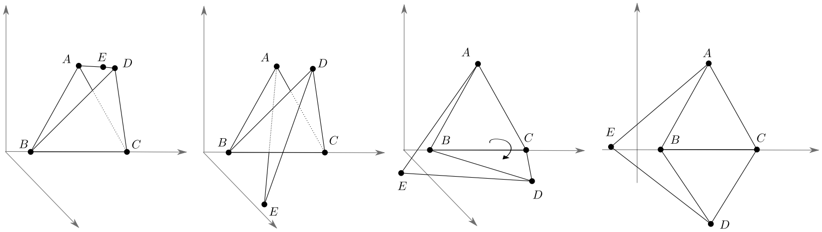

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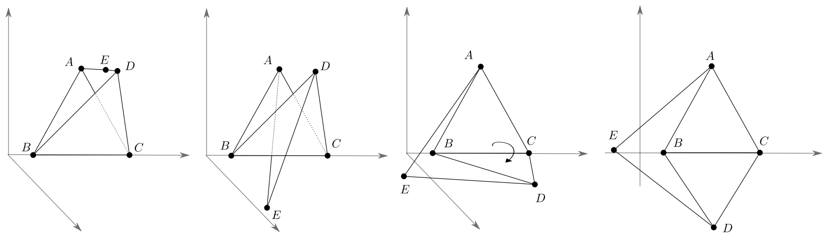

Theorem

G is d-flattenable iff all generic frameworks of G are d-flattenable.

Proof is in paper, it uses the fact that the distance map is a closed map.

Revised Question

What can we say about G if $\exists$ a generic d-flattenable framework?

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Existence of a generic d-flattenable framework is equivalent to independence in the rigidity matroid!

Connections to Rigidity

Theorem

For general l_p norms, $\exists$ a generic d-flattenable framework of G iff G is independent in the d-dimensional generic rigidity matroid.

[6] D. Kitson, *Finite and infinitesimal rigidity with polyhedral norms*, 2014. eprint: arXiv:1401.1336.

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Theorem

For general l_p norms, $\exists$ a generic d-flattenable framework of G iff G is independent in the d-dimensional generic rigidity matroid.

For norms other than l_2, we can use the formulation of the rigidity matroid as well as notion of well-positioned frameworks of Kitson[6].

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Proof Idea

One direction ("if") of this result is actually a restatement from Asimow and Roth[7]

Theorem (Asimow and Roth)

Let $f : R^n \to R^m$ be a smooth map and $k = \max\{rank(df(x))\}$. If x_0 is a regular point of f, then the image under f of some neighborhood of x_0 is a k-dimensional manifold.

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In our case, f is the distance map from point configurations to length-vectors (which is smooth). Then the generic rigidity matrix is the Jacobian or df.

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Because G is independent, the rank in the d-dimensional rigidity matroid is $|E| = k$.

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So, our distance map will map some neighborhood of a framework to a $|E|$-dim manifold in Φ^d_{G,l_p}.

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But, this is exactly the same dimension it would map to in Φ_{G,l_p}. So, that framework is d-flattenable.

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Proof Idea

Theorem (Restatement)

For general l_p norms, $\exists$ a generic d-flattenable framework of G iff G is independent in the d-dimensional generic rigidity matroid.

One direction done due to Asimow and Roth, onto the "only if" part

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Theorem (Restatement)

For general l_p norms, $\exists$ a generic d-flattenable framework of G iff G is independent in the d-dimensional generic rigidity matroid.

If there exists a generic d-flattenable framework (G, r), then the corresponding distance vector δ_r has an open neighborhood Ω_r in the interior of Φ_{n,l_p} (consequently in Φ_{G,l_p}).

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Likewise, the d-flattening (G, S) and its distance vector δ_s form an open neighborhood Ω_s in the relative interior of Φ^d_{n,l_p} (Φ^d_{G,l_p}).

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Because Ω_s is in the interior of Φ^d_{G,l_p}, it has dimension $|E|$.

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Again, the rigidity matrix of G is the Jacobian of the distance map (as before). The distance map is defined by polynomials and because Ω_s has dimension $|E|$, these polynomials are algebraically independent.

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Theorem (Restatement)

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Hence the Jacobian has rank $|E|$ as well, meaning G is independent.

Quick Question

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- The notion of well-positioned allows us to treat the l_p^p polynomials described before as essentially linear.
- For polyhedral norms, well-positioned frameworks always have a neighborhood in which each framework is also well-positioned.

Further results

Corollary

For general l_p norms, a graph G is:

- *independent in the generic d-dimensional rigidity matroid iff Φ^d_{G,l_p} has dimension $|E|$.*
- *maximal independent iff the projection of Φ^d_{n,l_p} onto G preserves its dimension and it is equal to $|E|$.*
- *rigid in d-dimensions iff the projection of Φ^d_{n,l_p} onto G preserves its dimension*

Open Problems

Conjecture

G is d-independent iff the projection of every face of Φ^d_{n,l_p} has dimension $|E|$.

Question

1. *Is d-flattening a continuous map over linkages*
2. *Is there a continuous path from high dimensional realization to d-dimensional realization for a d-flattenable linkage?*

Open Problems

Finally, the previous question of the forbidden minor characterization of 2-flattenability under l_1 remains open. Currently the biggest question is the wheel on 5 vertices.

Our Paper: M. Sitharam and J. Willougby, "On flattenability of graphs," in *Post-proceedings of ADG*, ser. LNAI, Springer, 2014

Bibliography

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