On Flattenability of Graphs

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July 12, 2015

Flattenability

A graph G is *d-flattenable* if for every linkage (bar-lengths in a given norm) that has a realization in some dimension also has a realization in R^d.[1]

Clearly d-flattenability is a minor closed property. So it has a forbidden minor characterization.

Under the l_2 norm, the only forbidden minor for 2-flattenability is K_4

Warm Up Question

What are the forbidden minors for 2-flattenability under l_1?

[1] M. Belk and R. Connelly, "Realizability of graphs," *Discrete Comput. Geom.*, vol. 37, no. 2, pp. 125–137, Feb. 2007.

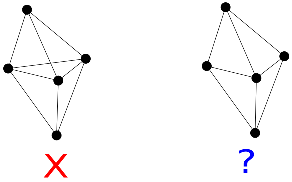
l_1 2-flattenability

K_4 is 2-flattenable under l_1[2]

For 5 vertex graphs, we have shown the following:

- K_5 is not 2-flattenable (Known)
- Banana is not 2-flattenable
- 4-Wheel is unknown (OPEN)
- All others are 2-flattenable

If 4-wheel is not 2-flattenable, it is the only forbidden minor



[2] H. Witsenhausen, "Minimum dimension embedding of finite metric spaces," *Journal of Combinatorial Theory, Series A*, vol. 42, no. 2, pp. 184–199, 1986.

Previous Work and Applications

This was shown in Sitharam-Gao[3] for the l_2 norm:

Theorem

For any l_p norm, a graph G is d-flattenable iff the set of attainable edge-length vectors for G in d-dimensions is convex [4].

Useful in many science or engineering applications.

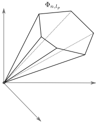
Proof makes extensive use of the cone of pairwise distance vectors.

[3] M. Sitharam and H. Gao, "Characterizing graphs with convex and connected cayley configuration spaces," *Discrete & Computational Geometry*, vol. 43, no. 3, pp. 594–625, 2010.

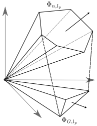
[4] Also called d-dimensional Cayley configuration space on G

Some Definitions

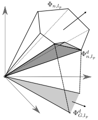
The *cone* [5] of all pairwise l_p^p-distance vectors on n-point configurations: Φ_{n,l_p}



The *d-dimensional stratum*: pairwise distance vectors of d-dimensional point configurations: Φ^d_{n,l_p}



The projection or shadow of this cone on an edge set G: Φ_{G,l_p}



Theorem (Restatement)

G is d-flattenable iff Φ^d_{G,l_p} is convex

[5] K. Ball, "Isometric embedding in lp-spaces," *European Journal of Combinatorics*, vol. 11, no. 4, pp. 305–311, 1990

Connections to Rigidity

Using the structure of Φ_{n,l_p}, we get a connection among:

- Flattenability
- Dimension of certain projections/strata of Φ_{n,l_P}
- Rigidity and Independence

For norms other than l_2, we can use the formulation of the rigidity matroid of Kitson[6].

Theorem

For general l_p norms, there exists a generic d-flattenable framework of G if and only if G is independent in the d-dimensional generic rigidity matroid.

[6] D. Kitson, *Finite and infinitesimal rigidity with polyhedral norms*, 2014. eprint: arXiv:1401.1336.

Open Problems

Conjecture

G is d-independent iff the projection of every face of Φ^{d}_{n,l_p} has dimension $|E|$.

Proof of another theorem raised another problem:

Question

1. *Is d-flattening a continuous map over linkages*
2. *Is there a continuous path from high dimensional realization to d-dimensional realization for a d-flattenable linkage?*

Open Problems

Can studying the Cayley configuration space of a certain class of graphs (partial 2-trees) lead to an extension of the Walker conjecture to partial 2-trees?

We may be able to better understand the entire structure of Φ^2_{n,l_2} by building it up from these partial 2-trees.

Our Paper: M. Sitharam and J. Willougby, "On flattenability of graphs," in *Post-proceedings of ADG*, ser. LNAI, Springer, 2014

Bibliography

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