Advances in Combinatorial and Geometric Rigidity (15w5114)

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1 Overview of the Field

The mathematical study of rigidity dates back at least to Maxwell's study of structures in the mid 19th century, when this still unsolved problem was implicitly posed: which graphs form rigid bar-and-joint frameworks in 3-space, for almost all configurations of the joints? With a revival of the mathematical and computational field of rigidity starting about 40 years ago, there has been an accelerating pace of interest and results, decade by decade [21]. In the last decade which included three prior 5-day BIRS workshops, broad areas of geometric and combinatorial rigidity have seen: (i) major growth and consolidation in the theoretical results and in the techniques being used; (ii) expansion of the range of theoretical and applied areas where the theory is being applied and where new problems have arisen; (iii) additional researchers joining the work representing a range of ages, including a new generation of researchers from mathematics and computer science emerging to become major contributors; (iv) an increased visibility as an active research area in more universities and more countries, and (v) a recognized source of techniques in new areas where there are geometric constraints. As one sign of this current activity, several major collective works from recent workshops should appear in the next months [6].

In simple terms, the geometric and combinatorial rigidity community focuses on multiple approaches for detecting whether an input set of polynomial equations representing a geometric constraint system generically (a) has a solution (independence); (b) has locally isolated solutions (rigid), or (c) has exactly one solution up to a space of "trivial" transformations in the chosen geometry (globally rigid). One summary of the questions that inspire current work in rigidity is:

–whether the properties listed above hold generically, and are therefore combinatorial, for appropriate forms of genericity?

– when the properties are combinatorial, whether there exists a polynomial time decidable combinatorial characterization for testing these properties, or a combinatorial characterization of a particular type, for example matroidal related to submodular counting properties, or by graph decompositions and inductive graph constructions?

– what changes in the geometry, metric, dimension etc. of an initial solution preserve the key rigidity properties?

– what is the impact of initial symmetry, or initial periodic repetitive structure, and what are the key geometric conditions for singular (non-generic) behavior which shifts independence to dependence, rigidity to flexibility, and global rigidity to multiple realizations?

2 Recent Developments and Open Problems

We outline recent progress and key questions in a selection of topics chosen through input from active researchers from recent BIRS workshops and Workshop [15w114].

2.1 Inductive Construction, Decomposition, Matroid methods and Efficient Algorithms for characterizing/recognizing (local and global) rigidity and independence in dimension 3 and higher.

These methods underpin core existing results, and provide examples and obstructions on the path towards solving the long standing open problem of combinatorial characterizations of the graphs which generically form rigid bar-and-joint frameworks in 3D and higher dimensions. There has been a revival of activity in this topic and this was the topic of our 2012 follow-on 2-Day BIRS workshop [3, 15, 8]. Even many known characterizations lack fast algorithms although the known cases of "pebble game" algorithms for sparse matroids and graded sparse matroids promise efficient extensions. As more computer scientists have entered the field, this topic is poised for a further surge of activity.

2.2 New developments on Global and Universal Rigidity

Unlike in the case of local rigidity, it was only recently shown that global rigidity is generic in all dimensions [5]. The algebraic techniques employed are exciting, and carry over to a related characterization of universal rigidity. The results and the techniques have recognized applications in a variety of settings from constructing superstable tensegrity structures to sensor network localization, and determination of molecular structures from NMR data (distances) as well as dimension reduction and compression. Yet, there is a sense that the new techniques from convex analysis and semidefinite programming for universal rigidity (global rigidity in all higher dimensions) have not been fully exploited. Nor are there sufficient efficiently algorithmic constructions to prove some of the conjectured results. It would be useful to study and develop examples and extensions which clarify a number of aspects of global rigidity and maybe decide for which problems global rigidity is the most appropriate concept (e.g. the concept of "stability" of protein and nano molecular structures).

2.3 Extensions and generalizations of existing results to other expanded types of constraints, objects and underlying geometries

So far, the results are overwhelmingly for distance constraints and some are restricted to the Euclidean geometry and Euclidean metric. Recently, most of these now classical results have been generalized, from bar-and-joint structures to body-bar frameworks in all dimensions and sometimes extended to results not available for bar-and-joint structures [21, 3], and to classes of distance and direction constraints [21, 7]. Some additional very recent developments include the determination of some key combinatorial results with applications to structures in non-Euclidean normed spaces [10], as well as to 3D bar-and-joint frameworks supported on 2-dimensional manifolds [12]. The search for generalizations is motivated not only by recognized applications in CAD, additional constraints for control of robotic formations (with direction constraints), and emerging applications such as machine learning and graphics (see Item 5), but also by a need to deepen/broaden the existing theory. Step by step this requires a unification of algebraic and geometric techniques including duality, coning, lifting and projections, projective and affine geometry and synthetic geometry.

2.4 Extensions of existing results to symmetric and periodic settings

(i.e., when the constraint systems, solutions, motions etc. are invariant under the action of various finite, periodic or crystallographic symmetry groups, and when their behavior changes from fully generic realizations).

One striking experience is that some symmetries can shift a generically rigid structure into a symmetry generic flexible structure and that these particular symmetries are common in protein structures (e.g. dimers for half-turn symmetry)[18] and periodic structures (e.g. inversive symmetry for crystals)[16]. Periodic structures have a mixed impact with some results that are more general than bar-and-joint frameworks [1, 15], and others which are more incomplete. There are starting to be algorithms that can test for this behavior under symmetry [17]. These are relatively new topics arising from crystallography (Zeolites and RUM Spectra [14]) and applications to material sciences such as the behavior of silica bilayers which show mirror symmetry between the layers. There has been an explosion of recent work and the connections to rigidity theory are still developing [6, 1].

2.5 Specific problems arising in various applications of independent mathematical interest

An example of this scenario is the recent work on a large repertoire of CAD constraints, using Grassmann-Cayley algebra for extracting geometric meaning within conditions under which efficient combinatorial methods can determine independence and sufficiency conditions for sparse graded graphs of an extended constraint system [11]. Following on prior work on direction-length constraints, each wider set of constraints opens up the full range of approaches: generic infinitesimal characterizations and fast combinatorial algorithms, the geometry of singularities, configuration spaces [19] and possible extensions to other metrics, inductive techniques, symmetry, global rigidity. Another example are results on independence, rigidity and global rigidity of certain types of incidence constraint systems promising provable bounds on a longstanding problem in Machine learning, called Dictionary learning [20]. More significantly, the above application examples require new types of constraints and geometries as well as a mix of existing and original approaches, that highlight rigidity theory's connections to diverse mathematical areas, and contribute to its increasing maturity and sophistication.

3 Presentation Highlights

We began the workshop with 5 minute presentations by all participants on problems they were working on and were looking for collaborations and feedback on. This established an immediate basis for engaged conversations from lunch on the first day on through to the last day when people were leaving, or were bridging into the companion 2 Day which started immediately after this workshop, with 25 of the participants.

A second key feature of our workshop was a substantial collection of model building materials which supported evening exchanges with sample models to encourage interdisciplinary communication, as well as a number of samples which were used to illustrate talks, including conjectures. This is a practice which our community has used effectively for our sequence of workshops, and one that supports lively evening conversations and development of conjectures, counter-examples, and alternative ways to place examples into various contexts for analysis.

We were able to support a sequence of survey talks which new-comers to the community report supported developing an initial overview of the state of the field as well as current unsolved problems. We then had a series of more focused presentations on recent results.

On the Thursday, an eminent Structural Engineer, William Baker from Skidmore, Owens, Merrill Ltd. (SOM), dropped by from a morning visit to a job site in Calgary to join conversations with four collaborators in a project to produce an annotated version of James Clerk Maxwell's 1864 (and following) works on reciprocal diagrams. On the same day, two of these collaborators presented related talks drawing on work of Maxwell as well as recent explorations of Structural Engineers interested in extending geometric tools to support insightful designs of buildings on all scales. We note that their work includes design and analysis for some of the tallest buildings in the world. These presentations and visits generated conversations and model building which extended late into the evening.

The opportunity to add on a 2 day workshop at the end of the 5 day workshop paid big dividends, by allowing developing collaborations to expand and consolidate. These collaborations will be a key theme below.

4 Scientific Progress Made

When asked for updates on the impact of the workshop, participants submitted the following brief descriptions. Overall, there is a strong desire for a future workshop in several years, and an appreciation for the impact of the two day extension which gave both an additional focus, and some further time to consolidate collaborations and progress, as mentioned below.

4.1 General responses

The following is typical of first-time participants Andy Vince: ”Impact, sure: As a relative newcomer to the area, the conference gave me a great overview of current research on combinatorial rigidity theory. ”

From an experienced participant (Patrick Fowler): The concentration of experts in a genuinely collaborative atmosphere made it hard *not* to have new ideas and get real work done. My thanks to BIRS and especially to the workshop organizers. I found the workshop very useful in getting an overview of the field and seeing the breadth of applications being made by mathematicians, engineers, computer scientists and physical scientists, meeting existing and potential collaborators and working with models brought along by other participants. The programme of talks and the opportunities for interaction with others gave me at least half a dozen directions for exploration in new research. The atmosphere was as always conducive to discussion, and time outside formal sessions was used to work intensively on a manuscript with two other participants.

Meera Sitharam reported that the graduate students in her research group benefited tremendously from the interaction with the larger international rigidity community. Walter Whiteley also reported a similar benefit for his Ph.D. student.

4.2 specific comments on collaborations and scientific progress

1. During the BIRS workshop, Patrick Fowler, Simon Guest and Bernd Schulze had several intensive research discussions over evenings with models, regarding classes of over-braced but typically flexible body-hinge frameworks. These structures are based on polyhedra with rigid faces where an independent subset of faces has been replaced by a set of holes.

   These symmetry calculations detect flexibility in these structures that is not revealed by counting alone. At the moment, we are exploring the rigidity and flexibility properties of symmetric ‘block-and-hole structures’ further. The process of revising the associated manuscript (see below) itself crystallised our thoughts on some further applications of symmetry to block-and-hole polyhedra, and a second ms is now in the early stages of preparation.

2. During the Banff workshop, Walter Whiteley, Yaser Eftekhari, Tony NixonShin-ichi Tanigawa and Bernd Schulze continued discussing the transfer of rigidity between spherical bar-joint frameworks (with points on the equator) and Euclidean point line frameworks, as well as connections to slider constraints from mechanical engineering and other recent rigidity work. The talk by Bill Jackson on Wednesday included examples that were quickly recognized as connected to this work and the discussion group broadened to include Bill.

   These explorations continued into the weekend, on the side of the Global Rigidity Workshop. At one point on Sunday, an ad hoc discussion included about 20 interested participants making connections among previously disjoint research results. This work has continued during various follow-up exchanes over the last few months. In the outcomes section we mention some manuscript(s) flowing from this ongoing work.

   A chapter on these connections is also proposed for the Handbook on Geometric Constraints, mentioned below.

3. Derek Kitson and Katie Clinch have been writing up a result regarding the existence of grid-like isostatic placements for graphs with specified edge-disjoint spanning trees. This work was initiated at the workshop.

4. Following observations during the workshop, Whiteley made some suggestions to Kitson after the workshop on direction length frameworks with a non-Euclidean norm. Kitson and Whiteley each made some notes and they will pick this up again later when time is available. Katie Clinch might like to be involved, having worked on both topics.

5. Bryan Chen and Derek Kitson found a corollary of the recent characterization by Cruickshank, Kitson and Power of the generic minimal rigidity of triangulated spheres with a single block (or hole). They have proven that the (3,6)-tightness condition on the ”block and hole graph” (of a triangulated sphere with a single block or hole) is equivalent to (3,0)-tightness of a graph constructed from its ”face graph” (terminology from the paper of Cruickshank, Kitson an Power). One advantage of the (3,0)-tightness condition is that it can be checked via the standard pebble game.

6. Wayne Lam and Bryan Chen have shown that the infinitesimal flexes of a certain family of periodic bar-joint frameworks (the ”twisted kagome” graphs) solve discrete versions of the Cauchy-Riemann equations and thus provide a realization of discrete complex analysis in the setting of rigidity theory. This is inspired by earlier work by the physicists Sun, Souslov, Mao and Lubensky on conformal invariance in the continuum elasticity of such frameworks.

7. Research interactions at the workshop by Meera Sitharam and Brigitte Servatius settled a problem about body-pin structures in 2D (where multiple bodies meet at a pin), partially with the help of an old theorem by Walter Whiteley.

8. Based on interactions at the workshop, Andy Vince and Meera Sitharam improved a previous algorithmic characterization of an abstract rigidity matroid and completed the proof that it gives the best known upper bound on the rank of the 3D rigidity matroid.

9. During the workshop, revisions were made to three draft chapters for the Handbook of Discrete and Computational Geometry by Schulze and Whiteley (Symmetry and Rigidity, and Rigidity and Scene Analysis) and Jordan and Whiteley (on Global Rigidity). One of these is now submitted, and the other two are in next to final drafts.

10. Steven Gortler had conversations with Anthony Man-So at BIRs about the relationship between a max rank PSD equillbrium stress matrix for a universally rigid framework, and the concept of “singularity degree” in the field of semi-definite programming (the topic of Anthony’s talk). See the outcomes section for the resulting publication.

11. Ideas that were discussed between Bob Connelly, Steven Gortler and Louis Theran at BIRS were instrumental to their recent (not yet published proof) that every graph that is generically globally rigid in R^d must have a generic d-dimensional framework that is universally rigid. This was an open question from the previous BIRs rigidity workshop.

12. The discussions of reciprocal diagrams resulted in continuing exchanges among four of the participants at the workshop. These will be followed up both in revisions to a Handbook Chapter mentioned below, and in a proposed chapter for the Handbook of Geometric Constrains mentioned in the next section.

    William Baker, a Civil Engineer from SOM submitted the following questions related to the rigidity of structures.

    Here is a list of things that structural engineers need to know that your group probably already knows.

    (a) How to tell when a 2D truss is stable or has mechanisms and how to find the mechanisms graphically.
    (b) An understanding of infinitesimal and finite mechanisms in 2D.
    (c) How quickly do infinitesimal mechanisms lock-up as they distort from the original geometry.
    (d) The effect of pre-stress on mechanisms not associated with the pre-stress in 2D.
    (e) How to tell when a 3D truss is stable or has mechanisms and how to find the mechanisms graphically.

(f) An understanding of infinitesimal and finite mechanisms in 3D.

(g) The effect of pre-stress on mechanisms not associated with the pre-stress in 3D.

(h) What is the higher dimension version of Maxwells 2D reciprocal diagrams and polyhedral (i.e. 3D trusses and 4D polytope (polychora?), etc.).

5 Outcome of the Meeting

The following concrete outcomes have been reported..

1. A manuscript on 'Mobility of a class of perforated polyhedra' (Patrick Fowler, Simon Guest, Bernd Schulze) was drafted during the meeting, completed shortly afterwards and accepted by International Journal of Solids and Structures in January 2016.

2. Conversations mentioned on above on singularity degree lead Steven Gortler to realize that recent results due to Connelly and Gortler on second order universal rigidty might help characterize singularity degree. These ideas now appear in [2].

3. The workshop was crucial in initiating an NSF-Applied Math proposal on "Stability of Structures Large and Small", by 4 of the US participants, (Robert Connelly, Steven Gortler, Mike Thorpe, and Meera Sitheram) If awarded, will help partially support research as well as outreach meetings of this community.

4. The workshop was key in consolidating the chapter topics and authors for the new Handbook of Geometric Constraint Principles being edited by 3 members of this community, for CRC press. Most of the authors were at the meeting, and outlines were generated during and shortly after the meeting. See`http://tinyurl.com/HandbookGCSP-outline` for more details.

5. Other interactions at the workshop helped Meera Sitherams group at the University of Florida to (a) complete a paper that will appear in the journal Symmetry (concerning symmetries in sphere-based assembly configuration spaces), edited by another of the participants, Bernd Schulze.

6. Following some initial conversations in Banff, Shin-ichi Tanigawa, Viktoria Kaszanitzky and Bernd Schulze also started to work on the generic global rigidity of periodic frameworks (with a fixed lattice representation). A paper on this topic should be submitted soon to a high-level journal fairly soon.

7. The group of Eftkhari, Jackson, Schulze, Tanigawa and Whiteley are now completing a fourth (final?) draft of a paper that summarizes the first portion of our expanding findings connecting points on the equator on the sphere with lines in point-line configurations in the plane, and sliders. The paper includes a number of new or translated combinatorial results for classes of frameworks. The results also include extensions to point-hyperplane frameworks and points on the equator of hyperspheres. We expect to submit this paper to a high-level journal within the next few months. We are also working on a second paper.

   Other connections from this work are drafted as chapters to appear in the Ph.D. thesis of Yaser Eftekhara.

From the reports of the participants, this was an amazingly productive workshop. The collaborations and share continue, with a related follow up meeting will be held as an ICMS Workshop on "Geometric Rigidity Theory and Applications" in Edinburgh May 30-June 3, 2016 `http://icms.org.uk/workshop.php?id=383`. All of the organizers of this ICMS workshop were participants in this BIRS workshop.

Overall, there is a strong desire for a future workshop in several years, again with a two day extension to consolidate collaborations and give an additional focus. Our thanks to BIRS for your continuing financial support and strong organizational support.

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